RICE UNIVERSITY

On the approximation of the Dirichlet to Neumann map for high contrast two phase composites

by

Yingpei Wang

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE

Master of Arts

APPROVED, THESIS COMMITTEE:

Liliana Borcea, Noah G. Harding Professor, Chair
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Abstract

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Many problems in the natural world have high contrast properties, like transport in composites, fluid in porous media and so on. These problems have huge numerical difficulties because of the singularities of their solutions. It may be really expensive to solve these problems directly by traditional numerical methods. It is necessary and important to understand these problems more in mathematical aspect first, and then using the mathematical results to simplify the original problems or develop more efficient numerical methods.

In this thesis we are going to approximate the Dirichlet to Neumann map for the high contrast two phase composites. The mathematical formulation of our problem is to approximate the energy for an elliptic equation with arbitrary boundary conditions. The boundary conditions may have highly oscillations, which makes our problems very interesting and difficult.

We developed a method to divide the domain into two different subdomains, one is close to and the other one is far from the boundary, and we can approximate the energy in these two subdomains separately. In the subdomain far from the boundary,

the energy is not influenced that much by the boundary conditions. Methods for approximation of the energy in this subdomain are studied before. In the subdomain near the boundary, the energy depends on the boundary conditions a lot. We used a new method to approximate the energy there such that it works for any kind of boundary conditions. By this way, we can have the approximation for the total energy of high contrast problems with any boundary conditions.

In other words, we can have a matrix up to any dimension to approximate the continuous Dirichlet to Neumann map of the high contrast composites. Then we will use this matrix as a preconditioner in domain decomposition methods, such that our numerical methods are very efficient to solve the problems in high contrast composites.

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First of all, I would like to thank my advisor, Professor Liliana Borcea, for her support, guidance and so much time on me. She is a great mathematician and mentor who I am really admired. She introduced me this very interesting project and helped me work it out all the way. I will not have or finish this project without her assistance and guidance. I learned a lot when I am working with her and I am looking forward to future cooperation.

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Chapter 1

Introduction and background

This thesis focuses on the problems with high contrast coefficients. We will discuss some interesting properties of these problems and we are going to develop some efficient numerical methods. The solutions of these problems may vary very fast in some places of the domain. They are much more difficult to solve, theoretically and numerically, than general problems with smooth coefficients.

1.1 Problems in high contrast media

We are considering the following isotropic elliptic problem in the domain $D \in \mathbb{R}^2$

$$\nabla \cdot [\sigma(\mathbf{x}) \nabla u(\mathbf{x})] = 0, \qquad \text{in } D \tag{1.1}$$

with Dirichlet boundary condition

$$u(\mathbf{x}) = \phi(\mathbf{x}), \qquad \text{on } \partial D, \tag{1.2}$$

or with Neumann boundary condition

$$\begin{aligned} \sigma(\mathbf{x})\frac{\partial u(\mathbf{x})}{\partial \mathbf{n}} &= I(\mathbf{x}), \qquad \text{on } \partial D, \\ \int_{\partial D} I ds &= 0, \end{aligned} \tag{1.3}$$

where $\sigma(\mathbf{x})$ is the conductivity, $u(\mathbf{x})$ is the potential, $\mathbf{n}$ is the unit out normal to the boundary.

The coefficient $\sigma(\mathbf{x})$ has high contrast values, which means the solutions of the problem (1.1) may vary fast in some places. In order to present our ideas in this thesis, we will only focus on the two phase composites media, which has high conductive inclusions embedded into some smooth background matrix. We suppose that σ only takes two different values, 1 in the background matrix and ∞ in the inclusions.

In general, we need to solve the problem (1.1) with boundary condition (1.2) or (1.3). Because of the high contrast of coefficients, it is very difficult to directly solve this problem numerically. We are more interested in the approximation of the Dirichlet to Neumann (DtN) map first. The DtN map is defined as following

$$\begin{aligned} \Lambda : \quad & H^{1/2}(\partial D) \to H^{-1/2}(\partial D) \\ & \Lambda\phi = \sigma\frac{\partial u}{\partial \mathbf{n}} \end{aligned} \tag{1.4}$$

where u is the solution of (1.1) with boundary condition (1.2).

Then we will use the approximation of DtN map as a preconditioner for the numerical methods, such that we can solve this kind of problems numerically in some more efficient way.

1.2 Overview of previous work

In the work by Borcea [6], Borcea and Papanicolaou [8], Borcea, Berryman and Papanicolaou [7], they first gave a rigorous proof for the approximation of the Dirichlet to Neumann (DtN) and Neumann to Dirichlet (NtD) maps for a high contrast media. The idea is to find special trial functions for the variational problems and their dual problems of the elliptic equations. In this way, they can have upper and lower bounds for the energies for any smooth boundary condition. Considering the connections between the energies and the DtN or NtD maps, they can have approximations for these maps.

In their work, they develop a way to construct these trial functions for the Kozlov's model [21] in continuum high contrast media. In some way, they build connections between the high contrast media with a related discrete resistor network, and use the properties of this network to approximate the properties of the high contrast media. For example they use the discrete DtN or NtD maps of the resistor network to approximate the DtN and NtD maps of the continuous problems in high contrast media. However, these constructions highly depends on the geometrical features of the media, which may not be generalized very easily.

In their approximation, they divide the boundary of the domain into pieces and use a constant in each piece to approximate the potentials on this piece. It works well for boundary condition without highly oscillation, because the boundary condition now are like piecewise constants. The error generated by the approximation of the boundary condition will not influence the whole results. However, when the oscillation of the boundary condition getting higher, the error from the piecewise constants approximation of boundary condition will show up. We will take care of it in this thesis.

There are some other models for high contrast media, which will have more geometrical properties. We can also use the similar idea as the work of Borcea et. al., which is using discrete resistor networks to simulate the high contrast media. Since we have more assumptions on geometrical properties of the media, it becomes easier to generalize the approximation methods. In the following works, they are mostly interested in the transport properties of high contrast media. However we will be very easy to know the transport properties if we know the DtN or NtD maps, which means it will be more difficult to approximate the DtN or NtD maps.

Keller [20] first gave an approximation of the effective conductivity in medium containing a dense array of perfectly conducting spheres or cylinders. In this kind of model, there are more geometrical features than Kozlov's model [21] for continuum high contrast media. Since the media is periodic in some way, it is very enough to study a special local problem and use it to approximate the overall properties. More precisely, it is enough to study the local properties of a square cross section with one sphere or cylinder as Keller did in his paper [20].

This is actually the idea of homogenization theory, which connects the properties of heterogeneous media in different scales. However, the homogenization theory works well for studying properties for finite contrast media, it doesn't work so well when the contrast of the media goes to infinity.

This leads to the work of Berlyand and Kolpakov [4], Berlyand and Novikov [5], Berlyand, Gorb and Novikov [3] and many related work. In their work, they made the assumptions on the geometrical properties of the media that the distance between neighbor inclusions and the size of the inclusions are in different scales. In this case, they can localize the fluxes in some special places of the media. It is enough to analyze one problem locally to approximate the properties of the whole problem. In these work, they also use the variational principles and discrete resistor networks approximation like the methods introduced by Borcea et. al. However, it is more flexible and easy to implement because their geometrical assumptions. Novikov [22] also use the similar method to study the properties of the high contrast media for nonlinear problems.

1.3 Contribution and outline

It is very important to understand the Dirichlet to Neumann (DtN) or Neumann to Dirichlet (NtD) map for high contrast problems. As we mentioned before, it will be very easy to know the transport properties of the media after knowing the maps. In inverse problems, Calderon [1] suggested to use DtN or NtD maps to recover the coefficients σ. In other words, we almost have the whole information of the media by knowing the DtN or NtD maps.

The DtN or NtD maps are also useful in developing efficient numerical methods. For example, in the nonoverlapping domain decomposition methods, they are used to be preconditioners for the equations on the interface.

In this thesis, we focus on approximation of the DtN map for the high contrast two phase composites. In other words, we are going to approximate the energy for this problem with any boundary conditions, including boundary conditions without or with oscillations. We will show that there will be three important parts in the approximation for the energy. The first part comes from the network effect, which is studied in previous work, for example Borcea et. al. [7]. The second part comes from the tangential flux effect, this effect will always be there with or without inclusions. However it is relatively small for boundary condition without that much oscillation, and it is ignored in previous work. The third part is the resonance effect, which exists also because of the inclusions and the oscillation of the boundary conditions. We will see that it will be small when the boundary condition almost has no oscillation or has only very high oscillation.

Here is the outline of this thesis. In chapter 2, we will give the mathematical formulation for our problem and present the mainly results. We will discuss some interesting properties of the discrete resistor network. In chapter 3, we will split the problem into two parts, which are problems in the area close to and far from the boundary. Also we will use the existing results for approximation of energy inside the domain. In chapter 4, we will carefully analyze the problems in the area close to the boundary such that our method will take care of any oscillation boundary conditions. In chapter 5, we will summarize what we did in this thesis and make proposal about how to apply our approximation for the DtN map to develop efficient numerical methods.

Chapter 2

Mathematical formulation and results

In this chapter, we will first give the mathematical formulation of the problem in infinite high contrast composites. Then we will review some results on discrete network approximation. Later we will discuss two basic problems and the resistor networks related to our problems. At last, we will present our results in section 2.5.

2.1 Formulation of the problem

2.1.1 The infinite high contrast problem

Infinite high contrast composites are media embedded with perfect conducting inclusions. For problems in infinite high contrast media, the equation (1.1) with boundary condition (1.2) will have the following form

$$\begin{aligned} \Delta u &= 0, && \text{in } \Omega \\ u &= t_i, && \text{on } \partial D_i, i \in \mathcal{S} \\ \int_{\partial D_i} \frac{\partial u}{\partial n} &= 0, && \text{for all } i \in \mathcal{S} \\ u &= \psi, && \text{on } \partial D \end{aligned} \tag{2.1}$$

where D is the disk with radius $L = O(1)$ in $I\!R^2$ and $\psi \in H^{1/2}(\partial D)$ is the Dirichlet boundary condition. D_i are identical disk inclusions inside the domain which stand for the perfect conducting inclusions, and $\Omega = D \setminus \cup_{i\in\mathcal{S}}\overline{D_i}$ is the domain where the material is not so well conducting. n is the outside normal to the boundary ∂D_i.

$\mathcal{S}$ is an index set for all the inclusions in the domain D, with $|\mathcal{S}| = N$. The inclusions are densely spaced but not touching each other, and they do not touch the boundary ∂D either. Let $\mathcal{S}_B$ be the index set for the inclusions which are very close to the boundary, with $|\mathcal{S}_B| = N_B$. $\mathcal{S}_I = \mathcal{S} \setminus \mathcal{S}_B$ is the index set for inclusions inside the domain, with $|\mathcal{S}_I| = N_I$. See figure 2.1 for example

The solution of the problem (2.1) is the couple $(u, \mathcal{T})$, where u is the potential in Ω. And $\mathcal{T} = (t_1, t_2, \cdots, t_N)$ are the potentials on the inclusions, we don't know them before solving the problem (2.1). Sometimes we will say u is the solution of the problem (2.1) without mention $\mathcal{T}$.

From the appendix A.1, we see that the problem (2.1) is the Euler-Lagrange equation for the following minimization problem of the energy

$$\mathcal{E}(\psi) = \min_{\phi\in V} \frac{1}{2} \int_\Omega |\nabla\phi|^2, \tag{2.2}$$

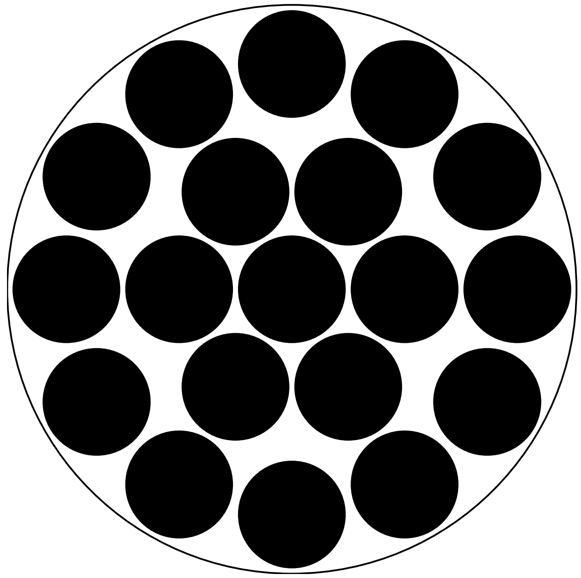


Figure 2.1: The domain with densely packed inclusions.

with the trial space

$$V = \{\phi \in H^1(\Omega) : \phi|_{\partial D} = \psi \text{ and } \phi|_{\partial D_i} = \text{constant } , \forall i \in \mathcal{S}\} \tag{2.3}$$

$\mathcal{E}(\psi)$ is the energy in the domain Ω of the problem (2.1) with boundary condition ψ. Also the minimizer of (2.2) satisfies the problem (2.1), which means

$$\mathcal{E}(\psi) = \frac{1}{2} \int_{\Omega} |\nabla u|^2, \tag{2.4}$$

when $(u, \mathcal{T})$ is the solution of the problem (2.1).

2.1.2 The Dirichlet to Neumann map

Our main purpose of this thesis is to approximate the DtN map of the problem (2.1), which is defined by

$$\begin{aligned} \Lambda : \quad & H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D) \\ & \Lambda\psi = \sigma \frac{\partial u}{\partial n} \end{aligned} \tag{2.5}$$

where u is the solution of the problem (2.1), and n is the outside normal to the boundary ∂D.

If $\psi \in H^{\frac{1}{2}}(\partial D)$ is the potential at the boundary, then $\Lambda\psi = \sigma\frac{\partial u}{\partial n} \in H^{-\frac{1}{2}}(\partial D)$ is the current flux at the boundary. We can define the following duality pairing

$$\langle \psi \, , \, \Lambda\psi \rangle := \int_{\partial D} \psi(\Lambda\psi) ds. \tag{2.6}$$

From the definition we can see that the DtN map Λ is self-adjoint and positive semidefinite, also see [24].

When $(u, \mathcal{T})$ is the solution for (2.1), we have

$$\begin{aligned} \langle \psi \, , \, \Lambda\psi \rangle &= \int_{\partial D} \psi(\Lambda\psi) ds = \int_{\partial D} \psi \frac{\partial u}{\partial n} = \int_{\partial D} \psi \frac{\partial u}{\partial n} - \sum_{i \in \mathcal{S}} t_i \int_{\partial D_i} \frac{\partial u}{\partial n} \\ &= \int_{\Omega} |\nabla u|^2 + \int_{\Omega} u \Delta u = \int_{\Omega} |\nabla u|^2 \\ &= 2\mathcal{E}(\psi) \end{aligned} \tag{2.7}$$

where $\mathcal{E}(\psi)$ is the energy defined in (2.2).

In other words, if we can approximate the energy $\mathcal{E}(\psi)$ of the problem (2.1) with any given boundary condition ψ, we can approximate the DtN map Λ for this problem.

For any constant boundary condition ψ, the solution for the problem (2.1) is the constant. It means the constant functions are in the null space of the DtN map Λ. We can add the constraint

$$\int_{\partial D} \psi ds = 0, \tag{2.8}$$

to the boundary condition ψ, which can deduce the null space of the DtN map Λ.

2.1.3 General high contrast problems

In this thesis, we will focus on the approximation of the DtN map for problems in infinite high contrast composites, but our approximation method will also work for problems in general high contrast composites

$$\begin{aligned} \nabla \cdot (\sigma_\epsilon \nabla u_\epsilon) = 0, &\qquad \text{in } D \\ u_\epsilon = \psi, &\qquad \text{on } \partial D \end{aligned} \tag{2.9}$$

where $\psi \in H^{\frac{1}{2}}(\partial D)$ is the Dirichlet boundary condition as before and

$$\sigma_\epsilon = \begin{cases} 1/\epsilon, & \text{in } D_i, \forall i \in \mathcal{S} \\ 1, & \text{in } \Omega \end{cases} \tag{2.10}$$

where $\epsilon > 0$ is a small positive parameter which reflects the contrast of coefficients.

For this problem we can also define the energy with boundary condition ψ as

$$\mathcal{E}_\epsilon(\psi) = \frac{1}{2} \int_D \sigma_\epsilon |\nabla u_\epsilon|^2$$

where u_ϵ is the solution of the problem (2.9).

Suppose the problems (2.1) and (2.9) have the same boundary condition ψ on the boundary ∂D, and let $\mathcal{E}(\psi), \mathcal{E}_\epsilon(\psi)$ denote the energy of these two problem separately. Bao,Li and Yin [2] gave the following approximation between the energy for general high and infinite high contrast problems

$$\mathcal{E}_\epsilon(\psi) = \mathcal{E}(\psi) + O(\epsilon). \tag{2.11}$$

In order to approximate the energy for a high contrast problem, we can approximate the energy for an infinite high contrast problem first.

Gorb found a first order corrector $\mathcal{F}$ and proved that

$$\mathcal{E}_\epsilon(\psi) = \mathcal{E}(\psi) + \epsilon\mathcal{F}(\psi) + o(\epsilon). \tag{2.12}$$

Also, she can improve this result into any higher order.

Calo, Efendiev and Galvis[12] proved the asymptotic expansion for the solution of high contrast problems to any order

$$u_\epsilon = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots . \tag{2.13}$$

Actually, if we extend the solution $(u, \mathcal{T})$ of (2.1) to the whole domain D by the constant t_i in each D_i, then we will have a solution u_0 in the whole domain D. Bao, Li and Yin [2] proved that as ϵ goes to 0, the solution u_ϵ for the problem (2.9) is weakly converged to u_0 in $H^1(D)$.

We can do similar approximation of the DtN map in any simple connected domain in $I\!R^2$ other then the disk, because any such domain can be mapped uniformly into an disk.

The inclusions can also have more general shapes other then small disks. Specially, they can be disks with different radii, which are in the same scale.

The coefficients can have more general form like

$$\sigma(x) = \sigma_0(x)\left(1 + \sum_{i\in\mathcal{S}}(1/\epsilon_i - 1)\chi_{D_i}(x)\right), \tag{2.14}$$

where $\sigma_0(x)$ is a smooth function and χ_{D_i} is the characteristic function. $\epsilon_i (i \in \mathcal{S})$ are small positive constants and they are not necessary to be the same.

2.2 The discrete resistor network approximation

Generally speaking, the discrete network approximation is a method to approximate some properties of the high contrast [7, 8] or infinite contrast [4, 5, 22] media by related properties of a discrete resistor network.

In this section, we will first review some definition and results about resistor network. Then we will give some simple examples of approximation in high contrast problems. At last we will show how to produce a resistor network from a high contrast composite.

2.2.1 The discrete resistor network

In this section, most definition and results are due to Curtis et. al. [14, 16, 15]. They summarized most results in the book [17].

Since we are discussing problems in two dimensions, we will only talk about planar networks in this section. A graph with boundary $G = (V, V_B, E)$ is a triple, where V denotes the set of all the nodes and $V_B \subset V$ denotes the set of nodes on the boundary. $E \subset V \times V$ denotes the set of edges. A planar graph is a graph G with boundary, which can be embedded into a disk in the plane such that the boundary nodes can be located at the boundary of the disk and other nodes can be located in the interior of the disk.

A resistor network is a pair (G, γ), where $\gamma : E \to I\!R^+$ is function that associates each edge $e_{ij} \in E$ of the graph G a positive conductance $\gamma(e_{ij}) = g_{ij}$. Here g_{ij} is the effective conductance for each edge in E. When we are going to use a resistor network to simulate the continuous high contrast media, we will show how to construct the discrete network and give explicit formulas for these g_{ij}.

The Kirchhoff matrix K of the resistor network (G, γ) is defined by

$$K_{ij} = \begin{cases} 0, & e_{ij} \notin E \\ -g_{ij}, & e_{ij} \in E \\ \sum_{k \neq i} g_{ik}, & j = i \end{cases} \tag{2.15}$$

The Kirchhoff matrix is symmetric and has all row sum zero. If it is necessary, we can write the matrix into the following blockwise form

$$K = \begin{bmatrix} K_{II} & K_{IB} \\ K_{BI} & K_{BB} \end{bmatrix}$$

where I is the index set for interior nodes which belong to $V \setminus V_B$, and B is the index set for boundary nodes in V_B.

Suppose there are N_I interior nodes and N_B boundary nodes. Let $\mathcal{T} \in I\!R^{N_I}$ be the potentials on all interior nodes, $\mathcal{J} = (\mathcal{J}_1, \mathcal{J}_2, \cdots, \mathcal{J}_{N_B})^T \in I\!R^{N_B}$ and $\mathcal{U} = (\mathcal{U}_1, \mathcal{U}_2, \cdots, \mathcal{U}_{N_B})^T \in I\!R^{N_B}$ be excitation currents and potentials on the boundary nodes. Where $\mathcal{J}$ also satisfies the condition

$$\sum_{k=1}^{N_B} \mathcal{J}_k = 0.$$

From the Kirchhoff law, we will have

$$\begin{bmatrix} K_{II} & K_{IB} \\ K_{BI} & K_{BB} \end{bmatrix} \begin{bmatrix} \mathcal{T} \\ \mathcal{U} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathcal{J} \end{bmatrix} \tag{2.16}$$

The discrete DtN map $\Lambda^D : I\!R^{N_B} \to I\!R^{N_B}$ for the discrete resistor network is defined by

$$\mathcal{J} = \Lambda^D \mathcal{U}, \quad \forall\, \mathcal{U} \in I\!R^{N_B} \tag{2.17}$$

The DtN map Λ^D is a $N_B \times N_B$ symmetric, positive and semidefinite matrix. The null space of Λ^D is the constant vectors. We can compute the DtN map from the Kirchhoff matrix

$$\Lambda^D = K_{BB} - K_{BI} K_{II}^{-1} K_{IB}. \tag{2.18}$$

Which means we will know the DtN map when we have the resistor network.

The energy of the discrete network with boundary potential $\mathcal{U}$ is defined by

$$\mathcal{E}^D(\mathcal{U}) = \frac{1}{2} \min_{\mathcal{T} \in V^D} \sum_{e_{ij} \in E} g_{ij} (t_i - t_j)^2 \tag{2.19}$$

with

$$V^D = \{\mathcal{T} \in I\!R^{N_I + N_B} : t_i = \text{constant } (i \in \mathcal{S}_I) \text{ and } t_i = \mathcal{U}_i (i \in \mathcal{S}_B)\}$$

It is proved in [7] that the minimizer of (2.19) will satisfy the equation (2.16), and we will have the following equality

$$\mathcal{E}^D(\mathcal{U}) = \frac{1}{2} \mathcal{U}^T \Lambda^D \mathcal{U}. \tag{2.20}$$

2.2.2 Extension of the discrete resistor network

In this section, we will discuss the relationship between the discrete DtN maps when we extend a resistor network to another one by some special way. The results in this section will be useful to simplify our final results for the approximation of the continuous DtN map.

Suppose there is a resistor network (G_1, γ_1), which has N_I interior nodes with index set $\mathcal{S}_I$ and N_B boundary nodes with index set $\mathcal{S}_B = \{1, 2, \cdots, N_B\}$. When $e_{ij} \in E_1$, suppose $\gamma_1(e_{ij}) = g_{ij}$. In this network, $e_{ij} \notin E_1$ when $i, j \in \mathcal{S}_B$. See Figure 2.2(a).

Then we connect the neighbors of the boundary nodes to get a graph G_2, and extend the function γ_1 to γ_2 such that it has definition on the new edges. Suppose that $\gamma_2(e_{ij}) = g_{ij}$ when $e_{ij} \in E_2 \setminus E_1$, and it has the same definition as γ_1 on edges belong to E_1. See Figure 2.2(b).

After that, we add N_B more nodes into the graph G_2, which become the new boundary nodes with index set $\mathcal{S}_{B'} = \{1', 2', \cdots, N'_B\}$. We add the new edges between the node $i \in \mathcal{S}_B$ and $i' \in \mathcal{S}_{B'}$. We can also extend the function γ_2 to γ_3, such that it has definition on the new edges. Suppose $\gamma_3(e_{ii'}) = g_i, (i \in \mathcal{S}_B)$, and it has the same definition as γ_2 on edges belong to E_2. See Figure 2.2(c).

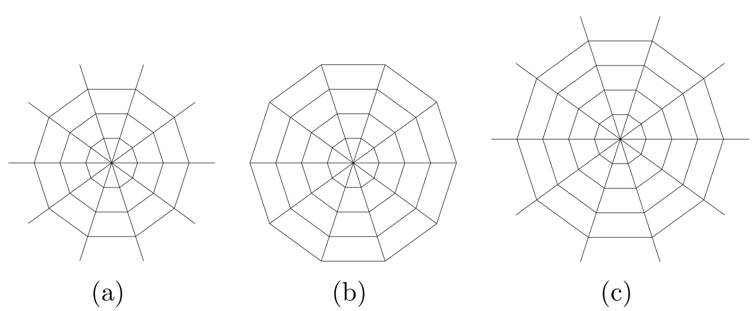


Figure 2.2: (a) The first network G_1, (b) The second network G_2, (c) The third network G_3.

Now suppose that the Kirchhoff matrix related to these three resistor networks are $K_1, K_2 \in I\!R^{(N_I+N_B)\times(N_I+N_B)}$ and $K_3 \in I\!R^{(N_I+2N_B)\times(N_I+2N_B)}$. Suppose the DtN maps related to these three resistor networks are $\Lambda_1^D, \Lambda_2^D, \Lambda_3^D \in I\!R^{N_B\times N_B}$.

Here are two useful propositions to simplify our result later in this thesis.

Proposition 2.2.1. *For a given vector $\mathcal{U} \in I\!R^{N_B}$, we have*

$$\mathcal{U}^T\Lambda_2^D\mathcal{U} = \mathcal{U}^T\Lambda_1^D\mathcal{U} + \sum_{e_{ij}\in E_2\setminus E_1} g_{ij}(\mathcal{U}_i - \mathcal{U}_j)^2.$$

Proof. Denote a new matrix $H \in I\!R^{N_B\times N_B}$ as following

$$H_{ij} = \begin{cases} 0, & i \neq j \text{ and } e_{ij} \notin E_2 \setminus E_1 \\ -g_{ij}, & i \neq j \text{ and } e_{ij} \in E_2 \setminus E_1 \\ \sum_{k\neq i} g_{ik}, & j = i \end{cases}$$

Then we only need to prove

$$\Lambda_2^D = \Lambda_1^D + H.$$

From the discussion in the last section, we can write K_1 into the blockwise form as

$$K_1 = \begin{bmatrix} K_{II} & K_{IB} \\ K_{BI} & K_{BB} \end{bmatrix}$$

From the formula (2.18), we have

$$\Lambda_1^D = K_{BB} - K_{BI}K_{II}^{-1}K_{IB}$$

From the struct of our resistor network, the matrix K_2 will have the following form

$$K_2 = \begin{bmatrix} K_{II} & K_{IB} \\ K_{BI} & K_{BB} + H \end{bmatrix}$$

Then

$$\Lambda_2^D = (K_{BB} + H) - K_{BI} K_{II}^{-1} K_{IB} = \Lambda_1^D + H.$$

$\square$

Proposition 2.2.2. *For a given vector $\Psi \in I\!R^{N_B}$, we have*

$$\Psi^T \Lambda_3^D \Psi = \min_{\mathcal{U} \in I\!R^{N_B}} \left\{ \mathcal{U}^T \Lambda_2^D \mathcal{U} + \sum_{i \in \mathcal{S}_B} g_i (\mathcal{U}_i - \Psi_i)^2 \right\}. \tag{2.21}$$

Proof. Denote the diagonal matrix $G \in I\!R^{N_B \times N_B}$ as

$$G = \mathrm{diag}\{g_1, g_2, \cdots, g_n\}.$$

Notice that Λ_2^D and G are both symmetric and Λ_2^D only has one zero eigenvalue. It is easy to prove that $\Lambda_2^D + G$ is a symmetric positive definite matrix.

We can write the right hand side of (2.21) as

$$\min_{\mathcal{U} \in I\!R^{N_B}} \left\{ \mathcal{U}^T (\Lambda_2^D + G) \mathcal{U} - 2 U^T G \Psi + \Psi^T G \Psi \right\},$$

which has minimizer $\mathcal{U}^* = (\Lambda_2^D + G)^{-1} G \Psi$ and then

$$\min_{\mathcal{U} \in I\!R^{N_B}} \left\{ \mathcal{U}^T (\Lambda_2^D + G) \mathcal{U} - 2 U^T G \Psi + \Psi^T G \Psi \right\} = \Psi^T (G - G(\Lambda_2^D + G)^{-1} G) \Psi$$

Then we need to prove

$$\Lambda_3^D = G - G(\Lambda_2^D + G)^{-1}G. \tag{2.22}$$

Suppose now we write K_2 into the blockwise form as

$$K_2 = \begin{bmatrix} K_{II} & K_{IB} \\ K_{BI} & K_{BB} \end{bmatrix}$$

From the formula (2.18), we have

$$\Lambda_2^D = K_{BB} - K_{BI}K_{II}^{-1}K_{IB}$$

From the struct of our resistor network, the matrix K_3 will have the following form

$$K_3 = \begin{bmatrix} K_{II} & K_{IB} & 0 \\ K_{BI} & K_{BB} + G & -G \\ 0 & -G & G \end{bmatrix}$$

Then

$$\begin{aligned} \Lambda_3^D &= G - \begin{bmatrix} 0 & -G \end{bmatrix} \begin{bmatrix} K_{II} & K_{IB} \\ K_{BI} & K_{BB} + G \end{bmatrix}^{-1} \begin{bmatrix} -G \\ 0 \end{bmatrix} \\ &= G - G(K_{BB} + G - K_{BI}K_{II}^{-1}K_{IB})^{-1}G \\ &= G - G(\Lambda_2^D + G)^{-1}G \end{aligned} \tag{2.23}$$

This proved (2.22). □

2.3 Some basic problems

2.3.1 The two disks problem

An interesting problem is to approximate the effective conductivity for periodic square lattices of disks, which is first discussed by Keller [20]. Keller derived the asymptotic formula of effective conductivity for a periodic spaced perfectly conducting disks embedded in an insulating background.

The mainly idea is that the current flux will only be strong in the area between closed space inclusions. When there are two perfectly conducting disks D_i, D_j embedded in the insulting background, there will be a neck Π_{ij} between these two disks,see Figure 2.3. The flux will be strong in the neck Π_{ij}. It is strong in the horizontal direction and it is almost linear.

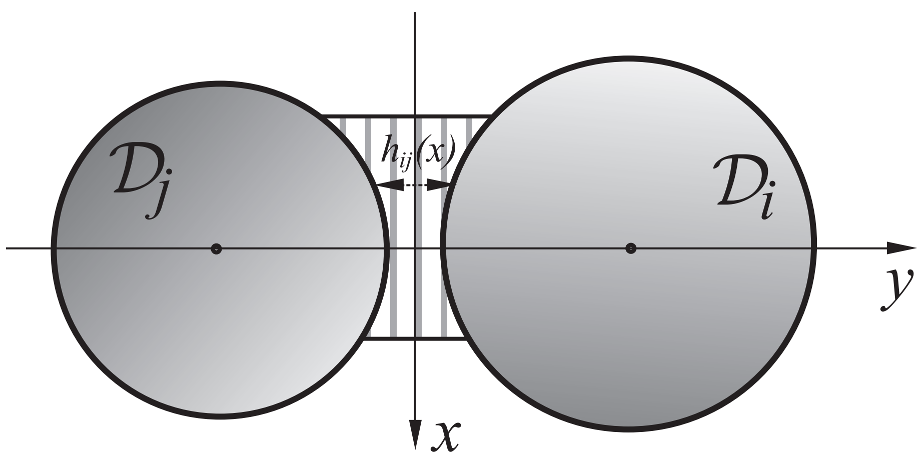


Figure 2.3: The neck Π_{ij} between D_i, D_j

Suppose the centers of these two disks are located at $O_i = (0, c)$ and $O_j = (0, -c)$, see Figure 2.3. The radius of the two disks are R_i, R_j, and the distance between them is δ_{ij}. We denote the up and bottom boundaries of the neck as $\partial\Pi_{ij}^{\pm}$, which are parallel to the line O_iO_j. The neck widths $S_{ij}^{\pm}$ are the distances from $\partial\Pi_{ij}^{\pm}$ to the line O_iO_j respectively.

Suppose the potentials are t_i, t_j on these two disks, separately. Then the partial differential equation yields this problem is

$$\begin{aligned} \Delta u &= 0, && \text{in } \Pi_{ij} \\ u &= t_i, && \text{on } \partial D_i \cap \partial \Pi_{ij} \\ u &= t_j, && \text{on } \partial D_j \cap \partial \Pi_{ij} \\ \frac{\partial u}{\partial n} &= 0, && \text{on } \partial \Pi_{ij}^{\pm} \end{aligned} \tag{2.24}$$

When u is the solution of the above equation, the energy in this neck is

$$\mathcal{E}_{\Pi_{ij}} = \frac{1}{2} \int_{\Pi_{ij}} |\nabla u|^2$$

From the appendix A.1, we see that

$$\mathcal{E}_{\Pi_{ij}} = \min_{\phi \in V_{\Pi_{ij}}} \left\{ \frac{1}{2} \int_{\Pi_{ij}} |\nabla \phi|^2 \right\},$$

with

$$V_{\Pi_{ij}} = \{ \phi \in H^1(\Pi_{ij}) : \phi|_{\partial D_i \cap \partial \Pi_{ij}} = t_i \text{ and } \phi|_{\partial D_j \cap \partial \Pi_{ij}} = t_j \}.$$

Any trial function in $V_{\Pi_{ij}}$ will give us a upper bound for $\mathcal{E}_{\Pi_{ij}}$.

For the lower bound, we need to use the Legendre transformation, see A.3,

$$\mathcal{E}_{\Pi_{ij}} = \max_{\mathbf{j} \in W_{\Pi_{ij}}} \left\{ t_i \int_{\partial D_i \cap \partial \Pi_{ij}} \mathbf{j} \cdot \mathbf{n} + t_j \int_{\partial D_j \cap \partial \Pi_{ij}} \mathbf{j} \cdot \mathbf{n} - \frac{1}{2} \int_{\Pi_{ij}} |\mathbf{j}|^2 \right\},$$

with

$$W_{\Pi_{ij}} = \{ \mathbf{j} \in \mathbf{L}^2(\Pi_{ij}) : \nabla \cdot \mathbf{j} = 0 \text{ and } \mathbf{j} \cdot \mathbf{n}|_{\partial \Pi_{ij}^{\pm}} = 0 \}.$$

Here $\nabla \cdot \mathbf{j} = 0$ is defined in the weak sense for $\mathbf{j} \in \mathbf{L}^2(\Pi_{ij})$.

Following the discussion in [4, 5], we will construct special trial functions like

$$\phi = \frac{1}{2}(t_i + t_j) + \frac{y}{h_{ij}(x)}(t_i - t_j) \text{ and } \mathbf{j} = (0, \frac{t_i - t_j}{h_{ij}(x)})^T \tag{2.25}$$

where $h_{ij}(x) = \delta_{ij} + (R_i - \sqrt{R_i^2 - x^2}) + (R_j - \sqrt{R_j^2 - x^2})$ is the distance between the left and right boundaries of the neck at height x.

For these two trial functions in (2.25), the upper and lower bounds given by the two variational formulas are very close. They only have $O(1)$ gap, which is relatively small comparing to the energy

$$\mathcal{E}_{\Pi_{ij}} = O(\sqrt{\frac{R}{\delta_{ij}}}) \gg O(1)$$

which we will see in (2.29).

Then the energy in the neck Π_{ij} is approximated by the upper or lower bound, which will end up with the following integration

$$\begin{aligned} \mathcal{E}_{\Pi_{ij}} &= \frac{1}{2}\int_{-S_{ij}^-}^{S_{ij}^+} |t_i - t_j|^2 \frac{dx}{h_{ij}(x)} + O(1) = \frac{1}{2}(t_i - t_j)^2 \int_{-S_{ij}^-}^{S_{ij}^+} \frac{dx}{h_{ij}(x)} + O(1) \\ &= \frac{1}{2} g_{ij}^0 (t_i - t_j)^2 + O(1) \end{aligned} \tag{2.26}$$

where

$$g_{ij}^0 = \int_{-S_{ij}^-}^{S_{ij}^+} \frac{1}{h_{ij}(x)} dx$$

is the effective conductance of the neck Π_{ij}.

If we suppose that inclusions are densely spaced $\delta_{ij} \ll \min\{R_i, R_j\}$, we will have an asymptotical approximation for g^0_{ij}, which is

$$g^0_{ij} = g_{ij} + O(1)$$

where

$$g_{ij} = \frac{\pi}{\sqrt{\delta_{ij}}}\sqrt{\frac{2R_iR_j}{R_i + R_j}}.$$

When the two disks have equal radii R, the effective conductance has approximation

$$g_{ij} = \pi\sqrt{\frac{R}{\delta_{ij}}}. \tag{2.27}$$

This approximation does not depend that much on the width $S^{\pm}$ of the necks, which means we have some freedom to choose the width of the necks. In general we can choose $S^{\pm}_{ij} \approx R/2$, such that we can give the above approximation easily. For more details, see [4, 5].

In our problem, when one of the disk is near the boundary, say D_i with radius R_i, suppose the distance between the disk and the boundary ∂D is δ_i now. Then the approximation effective conductance of this boundary neck will be

$$g_i = \pi\sqrt{\frac{2LR_i}{(L - R_i)\delta_i}}. \tag{2.28}$$

Here L is the radius of the disk domain D. In Chapter 4, we will show how to get this approximation in details.

In the work of Berlyand et. al. [4, 5], the boundary is straight there and they simulate the boundary as a quasi disk with radius ∞. Their formula for the effective conductance of the neck is an extreme situation when $L = \infty$ in the equation (2.28), which is

$$g_i = \pi \sqrt{\frac{2R_i}{\delta_i}}.$$

When we are trying to use a resistor network to simulate a high contrast media, we will see that g_{ij} and g_i will be the conductivity we assigned to a related edge in the resistor network. They are actually the approximation of the local effective conductance in a neck shape area in high contrast media.

So we have the following approximation for energy in a single neck

$$\mathcal{E}_{\Pi_{ij}} = g_{ij}(t_i - t_j)^2 + O(1) = g_{ij}(t_i - t_j)^2[1 + O(\sqrt{\frac{\delta_{ij}}{R}})] \tag{2.29}$$

which has the framework like the formula in (2.19). This formula actually separates the geometry property and physical property of the problem. Then we can associate an resistor with effective conductance g_{ij} for this neck. This is the mainly idea for the approximation of energy in our thesis, but it is much more complicated when there are more inclusions and oscillation boundary conditions.

Remark 2.3.1. *In the formula* (2.29)*, we prefer to absorb the error by the leading order term like we showed in the third term. For a single neck, there is no problem to write the error as* $O(1)$*. But when there are more necks in the domain and we need to add all the energy in the necks together, the third term in* (2.29) *will be more accurate.*

2.3.2 The problem with oscillation boundary condition

The second problem is an elliptic equation in homogenous media with oscillation boundary condition. We will see how does the boundary condition influence the energy of the problem.

Let's consider the following Laplace's equation

$$\Delta u = 0$$

in the disk

$$D = \{(r, \theta) : r < L, 0 \le \theta < 2\pi\}$$

with the boundary condition

$$u(L, \theta) = \cos k\theta,$$

where k reflects the oscillation of the boundary condition.

We consider this problem in the polar coordinate system. The solution of this problem is

$$u(r, \theta) = (r/L)^k \cos k\theta$$

and the flux in radius and tangential directions are

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{k}{L}(r/L)^{k-1} \cos k\theta, \\ \frac{\partial u}{\partial \theta} &= -k(r/L)^k \sin k\theta. \end{aligned}$$

Near the boundary, the tangential flux has almost the same important influence as the radius flux. From the discussion for the first problem in this section, we see that the neck approximation only consider the flux in one direction, but ignore the flux in the other direction. Which means we cannot just use a neck to simulate the parts of the domain near boundary when the boundary condition has highly oscillations.

However, the flux decays like $(r/L)^{k-1}$ when it goes away from the boundary. The energy in the whole domain is

$$\mathcal{E} = \frac{1}{2}\int_D |\nabla u|^2 = \frac{k\pi}{2}. \tag{2.30}$$

The energy in the boundary layer $\{(r,\theta) \in D : (1-\delta)L \leq r \leq L\}$ is

$$\begin{aligned} \mathcal{E}_\delta &= \frac{1}{2}\int_{(1-\delta)L}^{L} r dr \int_0^{2\pi} d\theta |\nabla u|^2 = \left(1-(1-\delta)^{2k}\right)\frac{k\pi}{2} \\ &= (1-(1-\delta)^{2k})\mathcal{E} \approx (2k\delta)\mathcal{E}. \end{aligned} \tag{2.31}$$

Which means the flux is mainly located in the boundary layer

$$\{(r,\theta) \in D : (1-\frac{1}{2k})L \leq r \leq L\}$$

The oscillation of the boundary will not influence too much far from the boundary. It suggests us to use discrete networks to simulate the parts far from the boundary, but not in the area near the boundary.

We can also define the Dirichlet to Neumann map for this problem like before, we denote it as Λ_1. We will have

$$\langle \cos k\theta \, , \, \Lambda_1 \cos k\theta \rangle = k\pi. \tag{2.32}$$

The discussion with boundary condition $u(L,\theta) = \sin k\theta$ will be similar, we will have

$$\langle \sin k\theta \, , \, \Lambda_1 \sin k\theta \rangle = k\pi. \tag{2.33}$$

It is also very easy to get

$$
\begin{aligned}
\langle \sin k\theta\,,\ \Lambda_1 \cos m\theta\rangle &= 0, \qquad \text{for all } k, m \\
\langle \cos k\theta\,,\ \Lambda_1 \cos m\theta\rangle &= 0, \qquad \text{for all } k \neq m \\
\langle \sin k\theta\,,\ \Lambda_1 \sin m\theta\rangle &= 0, \qquad \text{for all } k \neq m
\end{aligned}
\tag{2.34}
$$

These are result for homogeneous media, which has no inclusions at all. We will see later that, it always has these parts in related approximation for the DtN map Λ of high contrast composites problems. However, the results for Λ will have some more parts which are related to the inclusions, and we call it network effect. As discussed in the first problem of this section, the network effect will be in the order $O(\sqrt{\frac{R}{\delta}}) \gg 1$ with an $O(1)$ error. When $k\pi = O(1)$, there will no problem to ignore the effect because of the boundary oscillation.

2.4 Geometric setup and partition of the domain

In this thesis, we will divide the domain $\Omega = D \setminus \cup_{i\in\mathcal{S}}\overline{D_i}$ into several subdomains, such that we can use the discrete network approximation for the parts far from the boundary and we can do analysis near the boundary such that it will take care of the oscillation of the boundary condition.

2.4.1 Geometric setup of the problem

First of all, we need to have some assumptions on geometric properties of the domain in our problems. Remember that the radius of the disk D is $L = O(1)$. We suppose that the radii of all the inclusions are $R \ll L$. The radii of these inclusions are not necessary to be the same, however it must be in the same scale $O(R)$.

Later in this section, we will define neighbor inclusions. We say an inclusion near the boundary ∂D is a neighbor of the boundary and we call it a boundary inclusion. The distance of two neighbor inclusions are the closest distance between their boundary. What is more, we assume that the distance between any two neighbor inclusions D_i, D_j is $\delta_{ij} > 0$, and the distance between the boundary inclusions D_i and the boundary ∂D is $\delta_i > 0$. We assume that they are bounded up by some parameter δ, which satisfies $\delta \ll R$.

In the summary, we have three different scales in our problem

$$\delta \ll R \ll L. \tag{2.35}$$

In generally, we can set the radius of the domain L to be 1. We use L instead of 1 here because we want to make sure that everything in our results is right in the scale sense.

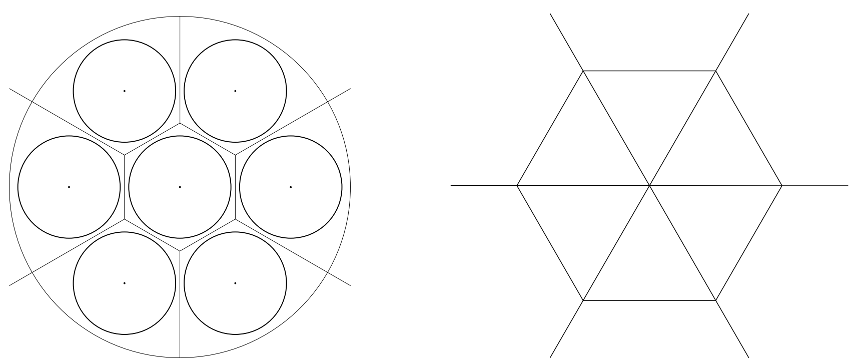
2.4.2 The partition of the domain

In order to give an exact definition of partition of the domain. We need to draw another circle D_ρ with radius $\rho = L - R/2$. From the assumptions of the scales in (2.35), we see that ∂D_ρ will intersect with any $\partial D_i (i \in \mathcal{S}_B)$ twice.

Remark 2.4.1. *It is not necessary for ρ to be exactly $1 - R/2$, it can be $1 - R/C$ for any reasonable constant $C > 1$. We only need to ensure that the circle ∂D_ρ will intersect with ∂D_i twice for any $i \in \mathcal{S}_B$.*

Now we are going to divide the domain $\Omega \cap \overline{D}_\rho$ into small pieces. The method of the dividing domains mainly comes from [4, 5, 22]. General speaking, we will divide $\Omega \cap \overline{D}_\rho$ into two parts. They are necks Π which will capture the mainly fluxes inside the domain, triangles Δ where the fluxes are weak and can be neglected.

In order to divide the domain, we first need to construct a discrete network from the high construct domain. The graph $G = (V, V_B, E)$ for the discrete network comes from the Delaunay triangulation of the domain D. The vertices V are centers of the disk inclusions, V_B are centers of inclusions near the boundary or nodes on the boundary ∂D. The edges E are edges of the Delaunay graph. The dual of Delaunay triangulation is the Voronoi tessellation, which can be used to define neighbors of the disks. When two disks share an edge of the Voronoi tessellation, they are neighbors in the network and there is an edge which connects them in Delaunay graph. A simple example is showed in Figure 2.4.

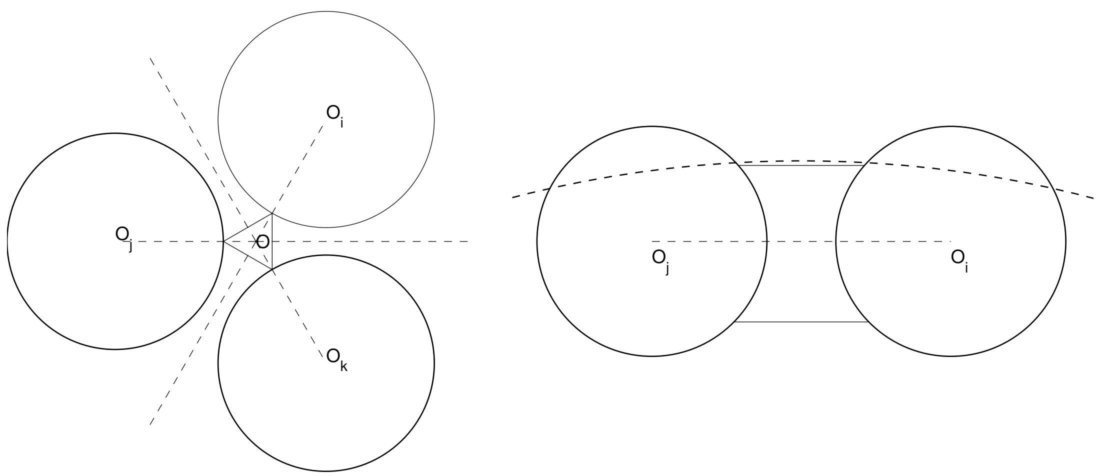


(a) (b)

Figure 2.4: Voronoi tessellation and Delaunay graph.

After we have the graph for discrete network, we can divide the domain D into different parts which are necessary for our discussion later. First we will describe how to construct triangles in $\Omega \cap \overline{D}_\rho$, the left parts in this domain will be necks. There are two different kinds of triangles in our problem. The first kind of triangles are located inside the domain $\Omega \cap \overline{D}_\rho$. Suppose a vertex O of the Voronoi tessellation is surrounded by three neighbor disks D_i, D_j, D_k with centers O_i, O_j, O_k separately. When we connect O with O_i, O_j, O_k, there will be one intersection on each circle. We use these three intersections as the vertice of the triangle we are going to construct. We denote this triangle as Δ_{ijk}, see Figure 2.5(a). Notice that each edge of the triangle Δ_{ijk} will parallel to O_iO_j, O_jO_k, O_kO_i respectively.

The second kind of triangles are located near the circle ∂D_ρ. Suppose the disks D_i, D_j are neighbors and $\partial D_i, \partial D_j$ have intersections with the circle ∂D_ρ. We can draw a straight line paralleled to the line O_iO_j, and it is as closed to ∂D_ρ as possible. In this way we have a small domain between the line and ∂D_ρ, we still call it a triangle and denote it as Δ_{ij}, see Figure 2.5(b).



(a) (b)

Figure 2.5: (a) The triangle Δ_{ijk} between the disks D_i, D_j, D_k. (b) Δ_{ij} between the dotted circle ∂D_ρ and the top solid straight line.

We denote the union of the two kinds of triangles as

$$\Delta := \bigcup \Delta_{ijk}, \text{ and } \Delta_B := \bigcup_{i,j \in \mathcal{S}_B} \Delta_{ij}, \tag{2.36}$$

The second kind of triangles are useful in the approximation for energy near the boundary. When the are N_B inclusions near the boundary ∂D, there are N_B such triangles in our problem.

The other parts in $\Omega \cap \overline{D}_\rho$ are necks. Each neck is located between neighbor disks and lies on an edge of the Delaunay graph. We denote the neck between disks D_i, D_j as Π_{ij}. We also divide the necks into two categories, although the methods for approximation energy in these necks are the same. One kind are necks which are neighbors of triangles in Δ_B, there are N_B such necks, we denote the union of them as

$$\Pi_B := \bigcup_{i,j \in \mathcal{S}_B} \Pi_{ij}, \tag{2.37}$$

and we denote the union of other necks as

$$\Pi := \bigcup_{i \notin \mathcal{S}_B \text{ or } j \notin \mathcal{S}_B} \Pi_{ij}. \tag{2.38}$$

We define the outside boundary layer as

$$B_0 = \Omega \setminus \overline{D}_\rho, \tag{2.39}$$

which is the domain between ∂D_ρ and ∂D in Ω.

We define the boundary layer in our problem as

$$B = B_0 \cup \Pi_B \cup \overline{\Delta}_B. \tag{2.40}$$

See Figure 2.6(a) and Figure 2.6(b).

In this way we can divide Ω into three different parts in our problem,

$$\Omega = B \cup \Pi \cup \overline{\Delta}.$$

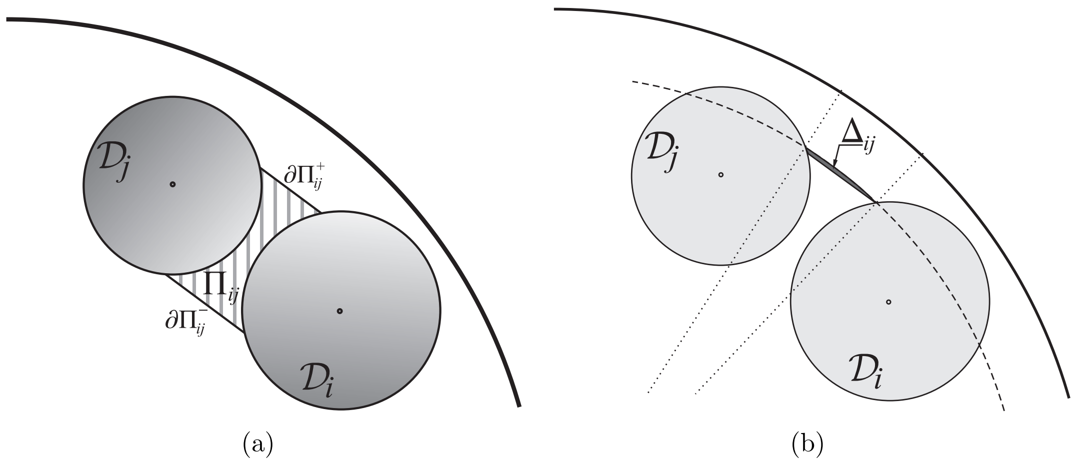


Figure 2.6: (a) The boundary neck in Π_B (b) The boundary triangle in Δ_B

See Figure 2.7(a).

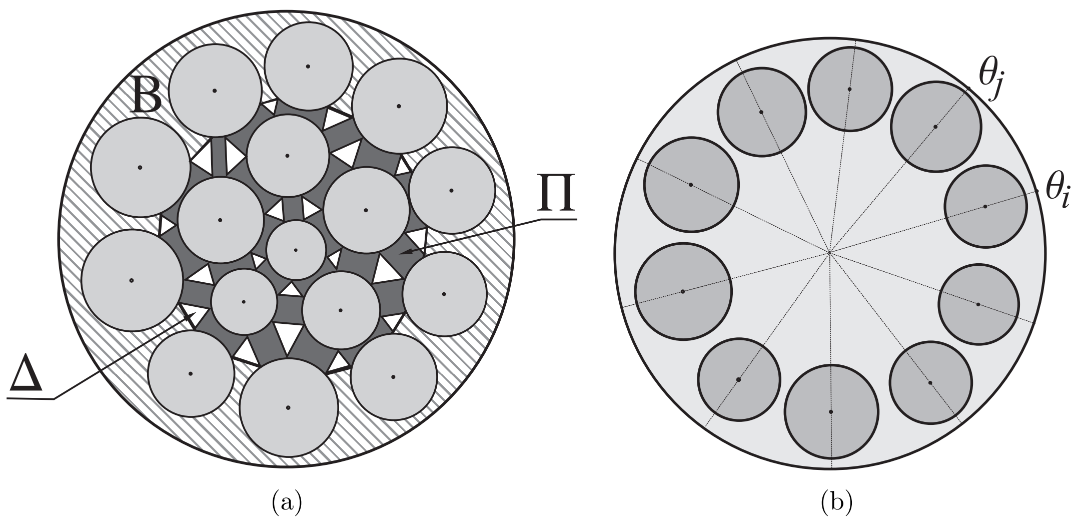


Figure 2.7: (a) The partition of the domain Ω. (b) Nodes on the boundary.

The effect of the boundary conditions oscillations will mainly locate in the boundary layer B. We will see the boundary layer B as a whole part in the discussion in Chapter 3.

In Chapter 4, we are going to discuss the details of the energy in B and we need to have partition of B. We already have the partition of B showed in (2.40), we also need some partition of the domain B_0. Remember that we draw another circle D_ρ

with radius $\rho = 1 - R/2$ to give exact definition before. We can also use this circle to help us divide B_0 into some subdomains which is useful in the analysis in Chapter 4.

Suppose the N_B inclusions $D_1, D_2, \cdots, D_{N_B}$ are very close to the boundary ∂D. The center of D_i is located at $O_i = (r_i, \theta_i)$, and the distance from D_i to ∂D is δ_i, see Figure 2.7(b). Then

$$r_i + R + \delta_i = L, \tag{2.41}$$

where $L = O(1)$ is the radius of the disk D. We also have the following assumption

$$\delta_i \leq \delta \ll R \ll L, \qquad \text{for all } i = 1, 2, \cdots, N_B \tag{2.42}$$

This ensures that the circle with radius ρ will intersect with each $\partial D_i (\forall i \in \mathcal{S}_B)$ twice.

For a special inclusion D_i, by connecting the origin $O = (0, 0)$ of the domain D and the two intersections,there will be two rays from the origin, which are $\theta = \theta_i \pm \alpha_i$ under the polar coordinate system, see Figure 2.8(a). Here the angle α_i satisfies

$$r_i^2 + \rho^2 - 2r_i\rho\cos\alpha_i = R^2 \tag{2.43}$$

which is uniquely determined by the position of the inclusion D_i.

For each D_i, denote the domain between these two rays and $\partial D_i, \partial D$ as B_i, see Figure 2.8(b). Which is

$$B_i = \{(r, \theta) : L - d(\theta) < r < L, \theta_i - \alpha_i < \theta < \theta_i + \alpha_i\} \tag{2.44}$$

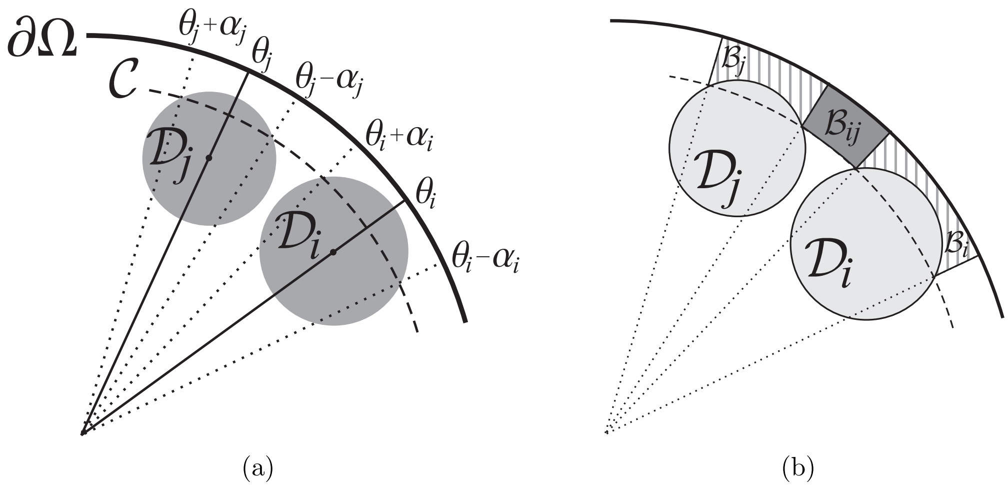


Figure 2.8: (a) The angles of the partition. (b) The partition of B_0.

where

$$d(\theta) = L - r_i \cos(\theta - \theta_i) - \sqrt{R^2 - (r_i \sin(\theta - \theta_i))^2} \tag{2.45}$$

with $\delta_i \leq d(\theta) \leq R/2$.

For neighbors D_i and D_j which are both close to the boundary ∂D, suppose $\theta_i < \theta_j$. Denote the area between B_i, B_j and $\partial D_\rho, \partial D$ as B_{ij}, see Figure 2.8(b). Which is

$$B_{ij} = \{(r, \theta) : L - d(\theta) < r < L, \theta_i + \alpha_i < \theta < \theta_j - \alpha_j\} \tag{2.46}$$

where $d(\theta) = R/2$ is a constant here.

The layer B_0 has the following partition

$$B_0 = \left(\bigcup_{i=1}^{N_B} B_i\right) \bigcup \left(\bigcup_{ij} B_{ij}\right) \tag{2.47}$$

2.4.3 The resistor networks related to our problem

In this section, we will associate two resistor networks to our problem. The first one is useful in the discussion, and the second one is useful for simplifying the results.

The first resistor network is (G_0, γ_0). Here the nodes of G_0 are the inclusions and the edges in G_0 are necks in Π introduced in Section 2.4. Notice that now the boundary nodes are centers of the boundary inclusions. For each e_{ij}, we have

$$\gamma_0(e_{ij}) = g_{ij},$$

where g_{ij} is the approximation for the effective conductivity of the neck $\Pi_{ij} \in \Pi$ introduced in Section 2.3. Suppose the discrete DtN map related to this resistor network is $\Lambda_0^D \in I\!R^{N_B \times N_B}$.

The second network (G, γ) is an extension of the first network, which has N_B more new boundary nodes and $2N_B$ more edges. The new nodes are located on the boundary ∂D of the domain and new edges will represent the necks $\Pi_{ij} \in \Pi_B$ and $B_i \in B_0$ with

$$\gamma(e_{ij}) = \begin{cases} \gamma_0(e_{ij}), & \Pi_{ij} \subset \Pi \\ g_{ij}, & \Pi_{ij} \subset \Pi_B \\ g_i, & B_i \subset B_0. \end{cases} \tag{2.48}$$

where g_{ij} is the approximation for the effective conductivity of the neck $\Pi_{ij} \in \Pi_B$. g_i tis he approximation for the effective conductivity of the neck $B_i \in B_0$, and we will show how to get this approximation in the analysis of Chapter 4. Suppose the discrete DtN map related to this resistor network is $\Lambda^D \in I\!R^{N_B \times N_B}$.

In this section, the extension of networks from (G_0, γ_0) to (G, γ) is the same as the extension of the networks from (G_1, γ_1) to (G_3, γ_3) in Section 2.2.2. We have the following equality from the discussion in Section 2.2.2,

$$\Psi^T \Lambda^D \Psi = \min_{\mathcal{U} \in I\!R^{N_B}} \left\{ \mathcal{U}^T \Lambda_0^D \mathcal{U} + \sum_{\Pi_{ij} \subset \Pi_B} g_{ij} (\mathcal{U}_i - \mathcal{U}_j)^2 + \sum_{i \in \mathcal{S}_B} g_i (\mathcal{U}_i - \Psi_i)^2 \right\} \tag{2.49}$$

for any given vector $\Psi \in I\!R^{N_B}$. This proposition is useful to simplify our results in Chapter 4.

2.5 The mainly results

We are going to approximate the DtN map Λ for the high contrast problem (2.1) in $I\!R^2$. It only depends on the domain and the conductivities, but not on the boundary condition. However, in order to approximate it, we need to approximate the energy

$$\mathcal{E}(\psi) = \frac{1}{2} \langle \psi \, , \, \Lambda \psi \rangle = \frac{1}{2} \int_{\partial D} \psi \Lambda \psi$$

for any given boundary condition ψ in (2.1).

The discrete network approximation works for boundary condition without that much oscillation. It uses piecewise constants to approximate the boundary condition, see [7]. And it also uses some resistor network to simulate the high contrast media, which can approximate the energy generated by the flux along some necks, see the first example in Section 2.3.

From the second example in Section 2.3, the oscillation of the boundary condition will generate some flux which is not along but vertical to the necks. And it will also have some contribution on the approximation of energy, which is not considered in [7] and related papers. However, the oscillation will not have so much influence far from the boundary, which means we can still use the discrete network approximation in the domain far from the boundary.

In our problem, the approximation will have two parts. One part is the discrete network approximation in the domain far from the boundary. The other part is the approximation in the boundary layer, which is the domain near the boundary. At the end, we will combine these two approximation in our results.

2.5.1 Review of existing results

For boundary condition without that much oscillations, Borcea et. al. [7] gave rigorous proof for the asymptotic resistor network approximation for a general high contrast problem. They use Kozlov's model [21] in continuum high contrast media. They suppose that the conductivity of the high contrast media have the following form

$$\sigma(\mathbf{x}) = \sigma_0 e^{-S(\mathbf{x})/\epsilon}. \tag{2.50}$$

where $\epsilon > 0$ is a small positive parameter which reflects the contrast of the problem. They can assign a discrete resistor network to a high contrast media according to the geometric properties of the function $S(\mathbf{x})$, which is actually the function $\sigma(\mathbf{x})$. And use the DtN map for the discrete resistor network to approximate the DtN map of the high contrast media. They have the following important results in [7]

Lemma 2.5.1. *Consider the asymptotic limit $\epsilon \to 0$. For any potential $\psi(\mathbf{x}) \in H^{\frac{1}{2}}(\partial\Omega)$, it has*

$$(\psi, \Lambda^{\epsilon}\psi) = < \mathcal{U}, \Lambda^{D,\epsilon}\mathcal{U} > [1 + o(1)]. \tag{2.51}$$

The components of $\mathcal{U}$ *are given by*

$$\mathcal{U}_j = \psi(s_j), \tag{2.52}$$

where s_j *denotes the point on* $\partial\Omega$ *that is associated with the boundary node* $j \in \mathcal{S}_B$*. They are intersection of ridges of maximal conductivity with the boundary.*

For the approximation (2.51), the choice of the potentials $\mathcal{U}_j = \psi(s_j)$ does not take care of the oscillation of the boundary condition. The formular (2.51) may not be exact when the boundary condition has highly oscillations.

2.5.2 Our approximation with general boundary condition

We are going to approximate the DtN map for the high contrast two phase composites, which is background matrix with low conductivity densely embedded with high conductivity inclusions. It is a different model from the Kozlov's model for high contrast media in [21]. However, our methods of analysis can be easily modified to problems with Kozlov's model.

When the boundary condition has different oscillation rate, the solution for the problem (2.1) will have very different performance. This motivates us to use different methods to approximate the energy for different kinds of boundary conditions. When the domain is a disk, the boundary condition $\psi(\theta)$ is defined on $[0, 2\pi)$. We have the following Fourier's formula

$$\psi(\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k^c \cos k\theta + a_k^s \sin k\theta\right)$$

Where

$$a_k^c = \frac{1}{\pi}\int_0^{2\pi} \psi(\theta)\cos k\theta \text{ and } a_k^s = \frac{1}{\pi}\int_0^{2\pi} \psi(\theta)\sin k\theta,$$

If we suppose our media is grounded, we will always have $a_0 = 0$. It is enough to discuss the problems with boundary condition $\sin k\theta$ and $\cos k\theta$ for any positive integer k.

We will first approximate the inner products $\langle \cos k\theta\,,\, \Lambda \cos k\theta \rangle$ and $\langle \sin k\theta\,,\, \Lambda \sin k\theta \rangle$ for any positive integer k.

Theorem 2.5.2. *For a given boundary condition* $\cos k\theta$ *or* $\sin k\theta$*, we will have the following approximation*

$$\begin{aligned} \langle \cos k\theta\,,\, \Lambda \cos k\theta \rangle &= k\pi + \left((S_k \Psi_k^c)^T \Lambda^D (S_k \Psi_k^c) + \mathcal{R}_k \cdot \mathbf{1} \right) [1 + O(\sqrt{\frac{\delta}{R}})] \\ \langle \sin k\theta\,,\, \Lambda \sin k\theta \rangle &= k\pi + \left((S_k \Psi_k^s)^T \Lambda^D (S_k \Psi_k^s) + \mathcal{R}_k \cdot \mathbf{1} \right) [1 + O(\sqrt{\frac{\delta}{R}})] \\ \langle \sin k\theta\,,\, \Lambda \cos k\theta \rangle &= \left((S_k \Psi_k^s)^T \Lambda^D (S_k \Psi_k^c) \right) [1 + O(\sqrt{\frac{\delta}{R}})] \end{aligned} \tag{2.53}$$

where Λ *is the DtN map for the high contrast composite we are trying to approximate.* $\Lambda^D \in I\!R^{N_B \times N_B}$ *is the DtN map the discrete resistor network introduced in Section 2.4.3.* $S_k = \mathrm{diag}\{S_{k1}, S_{k2}, \cdots, S_{kN_B}\} \in I\!R^{N_B \times N_B}$ *is the decay matrix with*

$$S_{ki} = \exp\left[-k \sqrt{\frac{2R\delta_i}{L(L-R)}} \right]$$

where δ_i *is the distance between the boundary inclusion* D_i *and the boundary of the domain* ∂D*.* Ψ_k^c, Ψ_k^s *are vectors of boundary conditions at* N_B *fixed points defined as*

$$\Psi_k^c = (\cos k\theta_1, \cos k\theta_2, \cdots, \cos k\theta_{N_B})^T \text{ and } \Psi_k^s = (\sin k\theta_1, \sin k\theta_2, \cdots, \sin k\theta_{N_B})^T.$$

For the angles θ_i*, see Figure 2.8(a).*

$$\mathcal{R}_k = (\mathcal{R}_{k1}, \mathcal{R}_{k2}, \cdots, \mathcal{R}_{kN_B}) \in I\!R^{1 \times N_B}$$

is the resonance vector with $\mathcal{R}_{ki}$ is given by

$$\mathcal{R}_{ki} = \frac{\pi}{2}\sqrt{\frac{2LR}{(L-R)\delta_i}}\left\{\sqrt{\frac{2k\delta_i}{L\pi}}Li_{1/2}\left(\exp\left[-\frac{2k\delta_i}{L}\right]\right) - \exp\left[-2k\sqrt{\frac{2R\delta_i}{(L-R)L}}\right]\right\}.$$

If we are going to approximate the DtN map Λ by a matrix, these three formulas give approximation for entries in the diagonal, subdiagonal and superdiagonal of this matrix.

$k\pi$ is the result for the same duality pairing in homogeneous media, see the discussion in Section 2.3. It will blow up to infinity as k goes to infinity. It will always be there with or without inclusions.

Remark 2.5.3. *When k is small such that $kR \leq L$, $k\pi = O(1)$ has the same order of error generated by the network approximation. The decay matrix S_k can be simplified to an identity matrix and the resonance term $\mathcal{R}_k$ is also $O(1)$. In this case Theorem 2.5.2 can be simplified to the following form*

$$\begin{aligned}
\langle \cos k\theta\,,\, \Lambda \cos k\theta \rangle &= \left((\Psi_k^c)^T \Lambda^D \Psi_k^c\right)[1 + O(\sqrt{\frac{\delta}{R}})] \\
\langle \sin k\theta\,,\, \Lambda \sin k\theta \rangle &= \left((\Psi_k^s)^T \Lambda^D \Psi_k^s\right)[1 + O(\sqrt{\frac{\delta}{R}})] \\
\langle \sin k\theta\,,\, \Lambda \cos k\theta \rangle &= \left((\Psi_k^s)^T \Lambda^D \Psi_k^c\right)[1 + O(\sqrt{\frac{\delta}{R}})]
\end{aligned} \tag{2.54}$$

This is actually the result in (2.51).

In order to have the whole information of the DtN map, we also need the approximation for the following duality pairings when $k \neq m$

$$\langle \sin k\theta\,,\, \Lambda \cos m\theta \rangle, \quad \langle \cos k\theta\,,\, \Lambda \cos m\theta \rangle, \quad \langle \sin k\theta\,,\, \Lambda \sin m\theta \rangle$$

More results about these duality pairings are in Section 4.3, see Theorem 4.3.4,Theorem 4.3.6 and Theorem 4.3.7. From those theorems, we basically have approximation of all entries in the approximation matrix for DtN maps.

Chapter 3

Separation of the problem

In this chapter, we will first show that the perforated energy $\mathcal{E}_p(\psi)$ is a good approximation of the energy $\mathcal{E}(\psi)$ for an infinite high contrast problem. Then we will divide the perforated energy $\mathcal{E}_p(\psi)$ into two parts. One part is the energy inside the domain, which is studied before and we will present the results directly. The other part is the energy in the boundary layer, we will discuss it in details in the next chapter.

3.1 Variational principles and perforated domain

3.1.1 The primal and dual problems

From the appendix A.1, we see that to solve the problem (2.1) is equivalent to solve the following minimization problem

$$\mathcal{E}(\psi) = \frac{1}{2} \min_{\phi \in V} \int_{\Omega} |\nabla \phi|^2, \tag{3.1}$$

with the trial function space

$$V = \{\phi \in H^1(\Omega) : \phi|_{\partial D} = \psi, \phi|_{\partial D_i} = \text{constant }, \forall i \in \mathcal{S}\}. \tag{3.2}$$

In general, we are trying to approximate the energy $\mathcal{E}(\psi)$ for any given boundary condition ψ. From the formulation (3.1), any function in the trial space V will give $\mathcal{E}(\psi)$ an upper bound.

From the appendix appendix A.3, we can do a Legendre transformation to get the dual of the problem (3.1),

$$\mathcal{E}(\psi) = \max_{\mathbf{j} \in W} \left\{ \int_{\partial D} \psi \mathbf{j} \cdot \mathbf{n} - \frac{1}{2} \int_{\Omega} |\mathbf{j}|^2 \right\}, \tag{3.3}$$

with the trial space

$$W = \{\mathbf{j} \in \mathbf{L}^2(\Omega) : \nabla \cdot \mathbf{j} = 0, \int_{\partial D_i} \mathbf{j} \cdot \mathbf{n} = 0, \forall i \in \mathcal{S}\}. \tag{3.4}$$

The flux $\mathbf{j}$ in W need not to be continuous, and the derivative of $\mathbf{j}$ is in weak sense. Any trial function in W will give $\mathcal{E}(\psi)$ a lower bound.

In order to find trial functions to satisfy the first condition in W, we are not going to construct a divergence free function directly, but we let

$$\mathbf{j} = \nabla^{\perp} H,$$

for some function $H \in H^1(\Omega)$. Then it will be divergence free automatically.

However, it is difficult to construct a trial function in W which satisfies the conservation condition

$$\int_{\partial D_i} \mathbf{j} \cdot \mathbf{n} = 0 \qquad \text{for all } i \in \mathcal{S}.$$

We will see later that when the boundary condition has highly oscillation, the way to construct the trial functions near the boundary is different from the way in the inside necks. It is even more difficult to construct $\mathbf{j}$ there to satisfy the conservation condition.

We need some other methods to approximate the energy $\mathcal{E}(\psi)$ for the problem (2.1). The perforated medium approach introduced by Berlyand, Gorb and Novikov in [3, 22] will work very well here.

3.1.2 The perforated domain

In our problem, we set the perforated domain as

$$\Omega_p = B \cup \Pi = \Omega \setminus \overline{\Delta},$$

which does not include those triangles and it is a subdomain of Ω, see Figure 3.1.

The idea is that the fluxes in the triangles Δ is small and the energy in Δ is also small in the original problem (3.1). We will see that $E_P(\psi)$ defined below in (3.5) is a good approximation for the energy $\mathcal{E}(\psi)$ in (3.1).

We can define the perforated energy in the domain Ω_p as

$$\mathcal{E}_p(\psi) = \frac{1}{2} \min_{\phi \in V_p} \int_{\Omega_p} |\nabla \phi|^2, \tag{3.5}$$

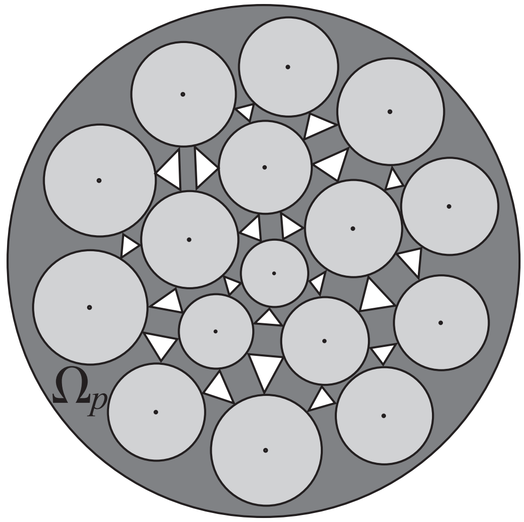


Figure 3.1: The perforated domain

where the trial space is smilar like (3.2)

$$V_p = \{\phi \in H^1(\Omega_p) : \phi|_{\partial D} = \psi, \phi|_{\partial D_i} = \text{constant }, \forall i \in \mathcal{S}\}, \tag{3.6}$$

This is a similar but a different minimization problem from (3.1). It is minimization problem defined in a smaller domain Ω_p.

In order to do analysis to the new problem (3.5), let us see the related differential equations first. From the appendix A.1, we see that minimizing $\mathcal{E}_p(\psi)$ over V_p leads to the following Euler-Lagrange equations:

$$\begin{aligned} \Delta u &= 0, && \text{in } \Omega_p \\ u &= t_i, && \text{on } \partial D_i, \forall i \in \mathcal{S} \\ \int_{\partial D_i} \frac{\partial u}{\partial n} &= 0, && \text{for all } i \in \mathcal{S} \\ \frac{\partial u}{\partial n} &= 0, && \text{on } \partial \Delta \\ u &= \psi, && \text{on } \partial D \end{aligned} \tag{3.7}$$

From the appendix A.2, the above equation has an unique solution. We would like to write the solution of this problem as $(u, \mathcal{U}, \mathcal{T})$, where $\mathcal{U} \in I\!R^{N_B}$ is the vector for potentials on the boundary inclusions and $\mathcal{T} \in I\!R^{N_I}$ is the vector for potentials on the inside inclusions.

3.2 The lower and upper bounds for $\mathcal{E}(\psi)$

In this section, we will first show that the energy $\mathcal{E}_p(\psi)$ defined in (3.5) is a lower bound for the energy $\mathcal{E}(\psi)$ defined in (3.1). Then we will show that $\mathcal{E}_p(\psi)$ is also a tight lower bound, hence it is a good approximation for the energy $\mathcal{E}(\psi)$.

3.2.1 The lower bound

Because Ω_P is a subdomain of Ω, we will have the following result

Lemma 3.2.1 (The lower bound)**.**

$$\mathcal{E}_p(\psi) = \frac{1}{2} \min_{\phi \in V_p} \int_{\Omega_p} |\nabla \phi|^2 \leq \frac{1}{2} \min_{\phi \in V} \int_{\Omega} |\nabla \phi|^2 = \mathcal{E}(\psi), \tag{3.8}$$

Proof. Suppose u is the solution of the minimization problem (3.1). Because Ω_p is a subset of Ω, $u|_{\Omega_p}$ is a trial function in V_p. Then

$$\mathcal{E}_p(\psi) = \frac{1}{2} \min_{\phi \in V_p} \int_{\Omega_p} |\nabla \phi|^2 \leq \frac{1}{2} \int_{\Omega_p} |\nabla u|^2 \leq \frac{1}{2} \int_{\Omega} |\nabla u|^2 = \mathcal{E}(\psi). \tag{3.9}$$

This means $\mathcal{E}_p(\psi)$ is a lower bound for $\mathcal{E}(\psi)$ for any given boundary condition ψ. □

3.2.2 The upper bound

Next we will find an upper bound for $\mathcal{E}(\psi)$ in (3.1). In Chapter 4, we will prove that there is a trial function $\phi_p \in V_p$ such that

$$\frac{1}{2}\int_{\Omega_p} |\nabla \phi_p|^2 = \mathcal{E}_p(\psi)[1 + O(\sqrt{\frac{R}{\delta}})], \tag{3.10}$$

Also from Chapter 4, we will see that the energy $\mathcal{E}_p(\psi)$ will be singular when the boundary condition ψ is not a constant

$$\mathcal{E}_p(\psi) = O(\sqrt{\frac{R}{\delta}}) \gg 1,$$

where R is the radius of the inclusions and δ is the distance between neighbor inclusions which satisfies the assumption (2.35).

We are going to extend ϕ_p from Ω_p to Ω, such that the extended function

$$\phi = \begin{cases} \phi_p, & \text{in } \Omega_p \\ \phi_\Delta & \text{in } \Delta \end{cases}$$

belongs to the trial space V in (3.2) and it will give us an upper bound for $\mathcal{E}(\psi)$.

We need the following Kirszbraun's theorem (see [19, 23]) to extend the trial function from the perforated domain Ω_p to the whole domain Ω.

Lemma 3.2.2 (Kirszbraun's theorem). *If $A, B \subset I\!R^m$ and $f : A \to I\!R^n$ is Lipschitzian, then f has a Lipschitzian extension $F : A \cup B \to I\!R^n$ with $Lip(F) = Lip(f)$.*

From the Lemma A.4.1, in order to make sure that the extend function ϕ belongs to $H^1(\Omega)$, we need to ensure that

$$||\gamma_0\phi_p - \gamma_0\phi_\Delta||_{\partial\Delta} = 0. \tag{3.11}$$

However, our trial function ϕ_p will be continuous in each subdomain (Π_{ij} or B) of Ω_p. The Lemma 3.2.2 says that the extended function ϕ will be Lipschitzian continuous. It will be continuous across $\partial\Delta$, so it satisfies the continuity condition in (3.11). A Lipschitzian function is differentiable almost everywhere, which means ϕ_Δ will belong to $H^1(\Delta)$. So the extended function ϕ will belong to $H^1(\Omega)$ automatically by applying the Lemma 3.2.2.

Another property of the extended function ϕ from the Lemma 3.2.2 is that it will keep the Lipschitzian constant. In other words,

$$|\nabla\phi_\Delta| \leq C \sup_{x\in\Omega_p} |\nabla\phi_p|.$$

But this bound is not so useful for us because $|\nabla\phi_p|$ will blow up in some place in Ω_p. However, we can use the value of $|\nabla\phi_p|$ near the boundary $\partial\Delta$ to bound $|\nabla\phi_\Delta|$, actually

$$|\nabla\phi_\Delta| \leq C \sup_{x\to\partial\Delta} |\nabla\phi_p|. \tag{3.12}$$

Suppose the triangle is Δ_{ijk}, whose vertices are located on the boundary of D_i, D_j, D_k respectively. It is surrounded by three necks Π_{ij}, Π_{jk} and Π_{ki}. Suppose the potentials on the three neighbor disks are t_i, t_j, t_k respectively. From the construction of the trial function ϕ_p in Section 3.4, we will see that there will be a constant C

which will not depend on R or δ such that

$$|\nabla\phi_p| \leq \frac{C}{R}\max\left(|t_i - t_j|, |t_i - t_k|, |t_j - t_k|\right) \quad \text{as } x \to \partial\Delta_{ijk} \tag{3.13}$$

Then the trial function ϕ satisfies

$$|\nabla\phi|_{\Delta_{ijk}} \leq \frac{C}{R}\max\left(|t_i - t_j|, |t_i - t_k|, |t_j - t_k|\right)$$

and we will have

$$\begin{aligned} \frac{1}{2}\int_{\Delta_{ijk}} |\nabla\phi|^2 &\leq (\frac{C}{R})^2 \max\left(|t_i - t_j|^2, |t_i - t_k|^2, |t_j - t_k|^2\right) \text{area}(\Delta_{ijk}) \\ &\leq C\max\left(|t_i - t_j|^2, |t_i - t_k|^2, |t_j - t_k|^2\right) \end{aligned} \tag{3.14}$$

where the constant C does not depend on R, δ.

From the formula (2.27), we see that

$$\frac{1}{2}\int_{\Pi_{ij}} |\nabla\phi|^2 = O(\sqrt{\frac{R}{\delta_{ij}}})|t_i - t_j|^2$$

It is similar in the necks Π_{jk} and Π_{ki}.

What we have is

$$\frac{1}{2}\int_{\Delta_{ijk}} |\nabla\phi|^2 = O(\sqrt{\frac{\delta}{R}})\max\left\{\frac{1}{2}\int_{\Pi_{ij}} |\nabla\phi|^2, \frac{1}{2}\int_{\Pi_{jk}} |\nabla\phi|^2, \frac{1}{2}\int_{\Pi_{ki}} |\nabla\phi|^2\right\}. \tag{3.15}$$

where δ is the upper bound for all δ_{ij}. It means the integration of the trial function ϕ in Δ will be small compared to the integration of ϕ in the necks Π.

Hence we have

$$\begin{aligned}\frac{1}{2}\int_{\Omega}|\nabla\phi|^2 = \frac{1}{2}\int_{\Omega_p\cup\overline{\Delta}}|\nabla\phi|^2 &= \frac{1}{2}\int_{\Omega_p}|\nabla\phi|^2 + \frac{1}{2}\int_{\Delta}|\nabla\phi|^2 \\ &= \frac{1}{2}\int_{\Omega_p}|\nabla\phi_p|^2[1+O(\sqrt{\frac{\delta}{R}})] = \mathcal{E}_p(\psi)\left[1+O(\sqrt{\frac{\delta}{R}})\right]^2 \\ &= \mathcal{E}_p(\psi)[1+O(\sqrt{\frac{\delta}{R}})]\end{aligned} \tag{3.16}$$

from (3.10). It means we find a trial function $\phi \in V$ and it will give an upper bound for $\mathcal{E}(\psi)$

$$\mathcal{E}(\psi) \leq \frac{1}{2}\int_{\Omega}|\nabla\phi|^2 \leq \mathcal{E}_p(\psi)[1+O(\sqrt{\frac{\delta}{R}})] \tag{3.17}$$

In the summary, we have

Lemma 3.2.3 (The upper bound)**.**

$$\mathcal{E}_p(\psi) \leq \mathcal{E}(\psi) \leq \mathcal{E}_p(\psi)[1+O(\sqrt{\frac{\delta}{R}})], \;\; as \;\; \frac{\delta}{R} \to 0 \tag{3.18}$$

for any given boundary condition ψ*.*

So $\mathcal{E}_p(\psi)$ is a good approximation for $\mathcal{E}(\psi)$ with any boundary condition ψ. In order to approximate $\mathcal{E}(\psi)$, it is enough to approximate $\mathcal{E}_p(\psi)$. In the next section, we will show how to separare the minimization problem (3.5) into a two lever minimization problem.

Remark 3.2.4. *An alternative way to prove that* $\mathcal{E}_p(\psi)$ *is a good approximation for* $\mathcal{E}(\psi)$ *is to extend the solution* u_p *of* (3.7) *from* Ω_p *to* Ω *directly. However, we need to discuss the properties of* u_p *near* $\partial\Delta$ *in* Ω_p*. It is not obvious for* u_p *to have the property as* ϕ_p *has in* (3.13)*.*

3.3 Separation of the perforated energy $\mathcal{E}_p(\psi)$

We the boundary condition ψ of the problem (3.5) has highly oscillation, there will be tangential flux in the boundary layer. However, the oscillation of boundary condition will not influence too much on the flux far from the boundary. The idea is to separare the energy $\mathcal{E}_p(\psi)$ into two parts, which are the energy inside the domain and the energy near the boundary. We will use different methods to approximate the energies in these two different parts.

Notice that the boundary layer B and the union of necks Π are disjoint, we can separate the minimization problem (3.5) into two parts like following

Lemma 3.3.1 (The first iterative minimization lemma)**.**

$$\begin{aligned}\mathcal{E}_p(\psi) &= \frac{1}{2}\min_{\phi\in V_p}\int_{\Omega_p}|\nabla\phi|^2 \\ &= \min_{\mathcal{U}}\left(\frac{1}{2}\min_{\phi\in V_B(\mathcal{U})}\int_B|\nabla\phi|^2 + \frac{1}{2}\min_{\phi\in V_\Pi(\mathcal{U})}\int_\Pi|\nabla\phi|^2\right)\end{aligned} \tag{3.19}$$

where

$$V_B(\mathcal{U}) = \{\phi\in H^1(B) : \phi|_{\partial D}=\psi, \phi|_{\partial D_i}=\mathcal{U}_i, \forall i\in\mathcal{S}_B\}.$$
$$V_\Pi(\mathcal{U}) = \{\phi\in H^1(\Pi) : \phi|_{\partial D_i}=\mathit{constant}\ , \forall i\in\mathcal{S}_I, \phi|_{\partial D_i}=\mathcal{U}_i, \forall i\in\mathcal{S}_B\}.$$

and $\mathcal{U}\in I\!R^{N_B}$ is a vector for potentials on inclusions which are adjacent to the boundary ∂D. $\mathcal{S}_B$ and $\mathcal{S}_I$ are index sets for inclusions near the boundary and inside the domain, respectively.

Proof. Define the energy in all the inner necks Π with $\mathcal{U}$ given as

$$\mathcal{E}_\Pi(\mathcal{U}) = \frac{1}{2}\min_{\phi\in V_\Pi(\mathcal{U})}\int_\Pi|\nabla\phi|^2 \tag{3.20}$$

where $\mathcal{E}_\Pi$ depends on the vector $\mathcal{U}$.

Minimizing $\mathcal{E}_\Pi(\mathcal{U})$ over V_Π leads to the following Euler-Lagrange equations:

$$\begin{aligned} \Delta u &= 0, \qquad \text{in } \Pi \\ u &= t_i, \qquad \text{on } \partial D_i, \forall i \in \mathcal{S}_I \\ \int_{\partial D_i} \frac{\partial u}{\partial n} &= 0, \forall i \in \mathcal{S}_I \\ u &= \mathcal{U}_i, \qquad \text{on } \partial D_i, \forall i \in \mathcal{S}_B \\ \frac{\partial u}{\partial n} &= 0, \qquad \text{on } \partial\Pi \cap \partial\Delta \end{aligned} \tag{3.21}$$

Also we can write the solution of the problem (3.21) as $(u, \mathcal{T})$, where $\mathcal{T} \in I\!R^{N_I}$ is the vector of potentials on the inside disks.

We can also define the erengy in the boundary layer B with ψ and $\mathcal{U}$ given as

$$\mathcal{E}_B(\psi,\mathcal{U}) = \frac{1}{2} \min_{\phi \in V_B(\mathcal{U})} \int_B |\nabla\phi|^2 \tag{3.22}$$

Minimizing $E_B(\psi,\mathcal{U})$ over V_B leads to the following Euler-Lagrange equations:

$$\begin{aligned} \Delta u &= 0, \qquad \text{in } B \\ u &= \mathcal{U}_i, \qquad \text{on } \partial D_i, \forall i \in \mathcal{S}_B \\ u &= \psi, \qquad \text{on } \partial D \\ \frac{\partial u}{\partial n} &= 0, \qquad \text{on } \partial B \cap \partial\Delta \end{aligned} \tag{3.23}$$

Now suppose $(u^*,\mathcal{U}^*,\mathcal{T}^*)$ is the unique solution of the problem (3.7), it is the minimizer of $\mathcal{E}_p(\psi)$ over V_p. We have

$$\mathcal{E}_p(\psi) = \frac{1}{2} \int_{\Omega_p} |\nabla u^*|^2 \tag{3.24}$$

Because both the problem (3.21) and (3.23) have unique solutions with given boundary conditions, see proof in appendix A.2. Then $u^*|_B$ is the unique solution for the problem (3.23) with boundary conditions ψ and $\mathcal{U}^*$, and $(u^*|_\Pi, \mathcal{T}^*)$ is the unique solution for the problem (3.21) with boundary condition $\mathcal{U}^*$. We have

$$\begin{aligned}
&\min_{\mathcal{U}} \left(\mathcal{E}_B(\psi,\mathcal{U}) + \mathcal{E}_\Pi(\mathcal{U})\right) \\
&\leq \mathcal{E}_B(\psi,\mathcal{U}^*) + \mathcal{E}_\Pi(\mathcal{U}^*) \\
&= \frac{1}{2}\int_B |\nabla u^*|^2 + \frac{1}{2}\int_\Pi |\nabla u^*|^2 \\
&= \frac{1}{2}\int_{\Omega_p} |\nabla u^*|^2 \\
&= \mathcal{E}_p(\psi)
\end{aligned} \tag{3.25}$$

On the other hand, for any given vector $\mathcal{U}$, suppose the solution of (3.21) is $(u_\Pi^*, \mathcal{T}^*)$ with $u_\Pi^* \in V_\Pi(\mathcal{U})$, and the solution of (3.23) is $u_B^* \in V_B(\mathcal{U})$. We can define a trial function v in V_p such that

$$v = \begin{cases} u_B^*, & \text{in } B \\ u_\Pi^*, & \text{in } \Pi \\ \mathcal{T}_i^*, & \text{on } \partial D_i (i \in \mathcal{S}_I), \\ \mathcal{U}_i, & \text{on } \partial D_i (i \in \mathcal{S}_B). \end{cases} \tag{3.26}$$

$v \in H^1(\Omega_p)$ is obvious because Π and B do not share any interface. So we have $v \in V_p$.

Then we will have

$$\begin{aligned}\mathcal{E}_p(\psi) &= \min_{\phi\in V_p} \frac{1}{2}\int_{\Omega_p} |\nabla\phi|^2 \\ &\le \frac{1}{2}\int_{\Omega_p} |\nabla v|^2 \\ &= \frac{1}{2}\int_{B} |\nabla u_B^*|^2 + \frac{1}{2}\int_{\Pi} |\nabla u_\Pi^*|^2 \\ &= \mathcal{E}_B(\psi,\mathcal{U}) + \mathcal{E}_\Pi(\mathcal{U}).\end{aligned} \tag{3.27}$$

And this is true for arbitrary vector $\mathcal{U}$, so we have

$$\mathcal{E}_p(\psi) \le \min_{\mathcal{U}} \left(\mathcal{E}_B(\psi,\mathcal{U}) + \mathcal{E}_\Pi(\mathcal{U})\right).$$

Then we proved

$$\mathcal{E}_p(\psi) = \min_{\mathcal{U}} \left(\mathcal{E}_B(\psi,\mathcal{U}) + \mathcal{E}_\Pi(\mathcal{U})\right). \tag{3.28}$$

□

Remark 3.3.2. *The left and right hand sides of* (3.28) *are two equivalent minimization problem. On the right hand side, if we combine the minimizers of the two first lever minimization problems* (3.22) *and* (3.20) *with the minimizer* $\mathcal{U}^*$ *of the second lever minimization problem, we will get the minimizer of the problem* (3.1) *on the left hand side. Hence it will satisfies the conservation law on the boundary disks*

$$\int_{\partial D_i} \frac{\partial u}{\partial n} = 0, \forall i \in \mathcal{S}_B$$

which are not appeared in the Euler-Lagrange equations (3.23) *and* (3.21).

Now we have separated the problem (3.5) into two problems, (3.20) and (3.22).

The two problems are in two disconnected domains, Π and B. The energies of the two problems both depends on $\mathcal{U}$. We can first approximate $\mathcal{E}_\Pi(\mathcal{U})$ and $\mathcal{E}_B(\psi,\mathcal{U})$ by some formulas of $\mathcal{U}$. Then solve the second step minimization problem over $\mathcal{U}$, which is a discrete optimization problem.

we need to approximate $\mathcal{E}_\Pi(\mathcal{U})$ and $\mathcal{E}_B(\psi,\mathcal{U})$ for any given vector $\mathcal{U}$. In the next section, we will use the discrete network approximation to estimate $\mathcal{E}_\Pi(\mathcal{U})$, which will only depends on $\mathcal{U}$ but not on the boundary condition ψ. And in Chapter 4, we will use variational principles to approximate $\mathcal{E}_B(\psi,\mathcal{U})$, which will depends on $\mathcal{U}$ and the boundary condition ψ.

3.4 Approximation in necks

We will use discrete network approximation to estimate $\mathcal{E}_\Pi(\mathcal{U})$. Borcea et al [8, 6, 7] gave very rigorous proof for this approximation for general high contrast problems. Berlyand et al [4, 5] discussed the densely packed composites problems very carefully. Novikov [22] discussed the nonlinear case for the high contrast problems and introduced the perforated medium approach there.

The problem (3.20) is defined on a collection of necks, and these necks are separated by disks and triangles. We can use the discrete network approximation for this problem.

From Novikov [22], we have the following iterative minimization lemma. We call it the second iterative minimization lemma here.

Lemma 3.4.1 (The second iterative minimization Lemma).

$$\mathcal{E}_\Pi(\mathcal{U}) = \frac{1}{2}\min_{\phi\in V_\Pi}\int_\Pi |\nabla\phi|^2 = \min_{T\in V_0^D(\mathcal{U})}\sum_{\Pi_{ij}}\mathcal{E}_{\Pi_{ij}}|t_i - t_j|^2, \tag{3.29}$$

where $T = (t_1, t_2, \cdots)^T$ *is a vector for potentials on all the inclusions and*

$$V_0^D(\mathcal{U}) = \{T \in I\!R^{N_I+N_B} : t_i = \textit{constant } (i \in \mathcal{S}_I) \textit{ and } t_i = \mathcal{U}_i(i \in \mathcal{S}_B)\}. \tag{3.30}$$

The definition for energy in each neck Π_{ij} *is*

$$\mathcal{E}_{\Pi_{ij}} = \frac{1}{2} \min_{\phi \in V_{\Pi_{ij}}} \int_{\Pi_{ij}} |\nabla \phi|^2 \tag{3.31}$$

with

$$V_{\Pi_{ij}} = \{\phi \in H^1(\Pi_{ij}) : \phi|_{\partial D_i} = \frac{1}{2}, \phi|_{\partial D_j} = -\frac{1}{2}\}. \tag{3.32}$$

This lemma separates the minimization problem (3.20) into a two lever minimization problem. The first lever is the minimization problems (3.31), and we have approximation to $\mathcal{E}_{\Pi_{ij}}$ in Section 2.3, which only depends on the geometric property of the necks. The second lever is a discrete minimization problem and it can be solved very easily.

We will have the following asymptotic approximation

$$\begin{aligned} \mathcal{E}_\Pi(\mathcal{U}) &= \min_{T \in V_0^D(\mathcal{U})} \sum_{\Pi_{ij}} \mathcal{E}_{\Pi_{ij}} |t_i - t_j|^2 \\ &= \frac{1}{2} \min_{T \in V_0^D(\mathcal{U})} \sum_{\Pi_{ij}} g_{ij} |t_i - t_j|^2 [1 + O(\sqrt{\frac{\delta_{ij}}{R}})], \end{aligned} \tag{3.33}$$

where g_{ij} is an approximation for the effective conductance of the neck Π_{ij}.

Define the following approximation energy

$$\mathcal{E}_\Pi^0(\mathcal{U}) = \frac{1}{2} \min_{T \in V_0^D(\mathcal{U})} \sum_{\Pi_{ij}} g_{ij} |t_i - t_j|^2 \tag{3.34}$$

The right hand side of (3.34) is a minimization problem of a quadratic form with g_{ij} given. We can associate this minimization problem with a discrete resistor network (G_0, γ_0) introduced in Section 2.4.3.

From Section 2.2, we have

$$\mathcal{E}^0_\Pi(\mathcal{U}) = \frac{1}{2}\min_{T\in V_0^D(\mathcal{U})}\sum_{\Pi_{ij}} g_{ij}|t_i - t_j|^2 = \frac{1}{2}\mathcal{U}^T\Lambda_0^D\mathcal{U} \tag{3.35}$$

where Λ_0^D is the DtN map of the discrete resistor network associate with the high contrast composite introduced in Section 2.4.3.

Then we have the following approximation for the energy in all the necks Π

$$\mathcal{E}_\Pi(\mathcal{U}) = \frac{1}{2}\mathcal{U}^T\Lambda_0^D\mathcal{U}[1 + O(\sqrt{\frac{\delta}{R}})]. \tag{3.36}$$

with given potentials $\mathcal{U}$ on the boundary inclusions.

Chapter 4

Analysis in boundary layer

We also need to approximate $\mathcal{E}_B(\psi,\mathcal{U})$ near the boundary. Here let's state the problem again. For a given vector $\mathcal{U}$ and the boundary condition ψ. We want to approximate

$$\mathcal{E}_B(\psi,\mathcal{U}) = \frac{1}{2}\min_{\phi\in V_B(\mathcal{U})}\int_B |\nabla\phi|^2. \tag{4.1}$$

with

$$V_B(\mathcal{U}) = \{\phi \in H^1(B) : \phi|_{\partial D} = \psi, \phi|_{\partial D_i} = \mathcal{U}_i, i \in \mathcal{S}_B\}. \tag{4.2}$$

We are going to prove that we can also have an quadratic form of $\mathcal{U}$ for the approximation of $\mathcal{E}_B(\psi,\mathcal{U})$. However, the coefficients for the quadratic form here will depend on the boundary condition ψ now.

From the appendix A.1, the minimizer of the problem (4.1) is the solution of the following Euler-Lagrange equations

$$\begin{aligned} \Delta u &= 0, && \text{in } B \\ u &= \mathcal{U}_i, && \text{on } \partial B \cap \partial D_i, \forall i \in \mathcal{S}_B \\ \frac{\partial u}{\partial n} &= 0, && \text{on } \partial B \cap \partial \Delta \\ u &= \psi. && \text{on } \partial D \end{aligned} \tag{4.3}$$

In order to get the lower bound, we need to do a Legendre transformation (see appendix A.3)

$$\mathcal{E}_B(\psi, \mathcal{U}) = \max_{\mathbf{j} \in W_B} \left\{ \int_{\partial D} \psi \mathbf{j} \cdot \mathbf{n} + \sum_{i \in \mathcal{S}_B} \mathcal{U}_i \int_{\partial D_i \cap \partial B} \mathbf{j} \cdot \mathbf{n} - \frac{1}{2} \int_{\Omega} |\mathbf{j}|^2 \right\}, \tag{4.4}$$

with the space

$$W_B = \{\mathbf{j} \in \mathbf{L}^2(B) : \nabla \cdot \mathbf{j} = 0, \mathbf{j} \cdot \mathbf{n}|_{\partial B \cap \partial \Delta} = 0\} \tag{4.5}$$

where the derivative of $\mathbf{j}$ is in weak sense. Also we only need $\mathbf{j} \in \mathbf{L}^2(B)$, which means we can construct $\mathbf{j}$ from parts by parts.

Remark 4.0.2. *In order to make sure the condition* $\mathbf{j} \cdot \mathbf{n}|_{\partial B \cap \partial \Delta} = 0$ *satisfies, we have to put some necks* Π_B *into the domain* B*. It is easy to construct* $\mathbf{j}$ *in necks* Π_B *such that this condition satisfies. But it is complicated to construct* $\mathbf{j}$ *in* B_0 *to satisfy this condition. That is why we put some necks* Π_B *into the boundary layer* B*.*

In order to present our ideas, we will discuss the situation when the boundary consition is $\cos k\theta$ and approximate $\mathcal{E}_B(\cos k\theta, \mathcal{U})$ in the following several sections. Later we will discuss how to approximate $\mathcal{E}_B(\psi, \mathcal{U})$ with a general boundary condition ψ.

4.1 The upper and lower bounds for $\mathcal{E}_B(\cos k\theta, \mathcal{U})$

In this section, we will first discuss how to construct trial functions for upper and lowers bounds for $\mathcal{E}_B(\cos k\theta, \mathcal{U})$. We need to construct the trial functions separately in Π_B, Δ_B and B_0. Then we will prove that the upper and lower bound are very close under our construction, which means we can use either the upper or the lower bound as an approximation for $\mathcal{E}_B(\cos k\theta, \mathcal{U})$.

4.1.1 The upper bound for $\mathcal{E}_B(\cos k\theta, \mathcal{U})$

For the upper bound, we need to construct a function in $H^1(B)$ and satisfies constraints in (4.2). From the Lemma A.4.1, we see that in order to construct a trial function piece by piece such that the function still belongs to $H^1(B)$, we only need the functions of different pieces match each other on the interface in L^2 sense. Now we are going to construct the trial functions for the upper bound piece by piece.

Construction in Π_B

For a neck showed in Figure 2.3, the flux will be strong in the horizontal direction, but weak in the vertical direction. We construct the trial functions in the necks Π_{ij} for the upper bound like what we did in Section 2.3

$$\phi(x,y) = \frac{1}{2}(\mathcal{U}_i + \mathcal{U}_j) + \frac{y}{h_{ij}(x)}(\mathcal{U}_i - \mathcal{U}_j), \forall y \in (-\frac{h_{ij}(x)}{2}, \frac{h_{ij}(x)}{2}), x \in (-S_{ij}^-, S_{ij}^+) \quad (4.6)$$

Under this construction, the trial function ϕ matches the boundary conditions on the left and right boundary of the neck Π_{ij}. Also like the discussion in Section 2.3, we have the following approximation in the neck Π_{ij} when $\delta_{ij} \leq \delta \ll R$

$$\begin{aligned} \frac{1}{2}\int_{\Pi_{ij}} |\nabla\phi(x,y)|^2 &= \frac{1}{2}\int_{-S_{ij}^-}^{S_{ij}^+} \frac{dy}{h(y)}(\mathcal{U}_j - \mathcal{U}_i)^2 + O(1) \\ &= \frac{1}{2} g_{ij}(\mathcal{U}_j - \mathcal{U}_i)^2[1 + O(\sqrt{\frac{\delta_{ij}}{R}})]. \end{aligned} \tag{4.7}$$

where g_{ij} is the effective conductance of the neck Π_{ij} introduced in (2.27).

Construction in B_0

Next we are going to construct the trial functions in B_0. The flux in the layer B_0 is much more complicated, because we need to consider the flux in both the radius and tangential directions. However, the idea is still trying to find some ϕ such that

$$\Delta\phi \approx 0$$

in the domain B_0.

We usually cannot give an explicit form of ϕ such that $\Delta\phi$ is exactly zero, but we are trying to make $\Delta\phi$ to be small. The idea is to write ϕ into combination of the boundary conditions and the potentials $\mathcal{U}$, but the linear combination will not take care of the tangential flux very well. We construct the trial function in the layer B_0 like

$$\phi(r,\theta) = w_k(r,\theta)\cos k\theta + w(r,\theta)\mathcal{L}(\mathcal{U}) \tag{4.8}$$

where the weight functions w_k and w both depend on r and θ. We need to carefully construct them such that $\phi(r,\theta)$ showed in (4.8) will be a good approximation of the solution to $\Delta\phi = 0$ in B_0.

We can construct the weight functions like following

$$w_k(r,\theta) = \frac{(r/L)^k - (1-d(\theta)/L)^{2k} r^{-k}}{1-(1-d(\theta)/L)^{2k}}, \tag{4.9}$$

which satisfies

$$\begin{aligned}&\frac{\partial^2 w_k}{\partial r^2} + \frac{1}{r}\frac{\partial w_k}{\partial r} - \frac{k^2}{r^2} w_k = 0,\\ &w_k(L-d(\theta),\theta) = 0 \text{ and } w_k(L,\theta) = 1,\end{aligned} \tag{4.10}$$

and

$$w(r,\theta) = \frac{\ln(r/L)}{\ln(1-d(\theta)/L)} \tag{4.11}$$

which satisfies

$$\begin{aligned}&\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} = 0,\\ &w(L-d(\theta),\theta) = 1 \text{ and } w(L,\theta) = 0,\end{aligned} \tag{4.12}$$

$\mathcal{L}$ here is a function of the vector $\mathcal{U}$ defined as

$$\mathcal{L}(\mathcal{U}) = \begin{cases}\mathcal{U}_i, & \text{on } \partial D_i \cap \partial B_i,\\ (1-\ell_{ij}(\theta))\mathcal{U}_i + \ell_{ij}(\theta)\mathcal{U}_j, & \text{on } \partial\Delta_{ij} \cap \partial B_{ij}\end{cases} \tag{4.13}$$

where ℓ_{ij} is linear on θ

$$\ell_{ij}(\theta) = \frac{\theta - (\theta_i + \alpha_i)}{(\theta_j - \alpha_j) - (\theta_i + \alpha_i)} \tag{4.14}$$

which satisfies

$$\ell_{ij}(\theta_i + \alpha_i) = 0 \text{ and } \ell_{ij}(\theta_j - \alpha_j) = 1$$

See Figure 4.1(a) and Figure 4.1(b) for more details. $\mathcal{L}$ is piecewise defined, it is constants on ∂B_i and linear on ∂B_{ij}.

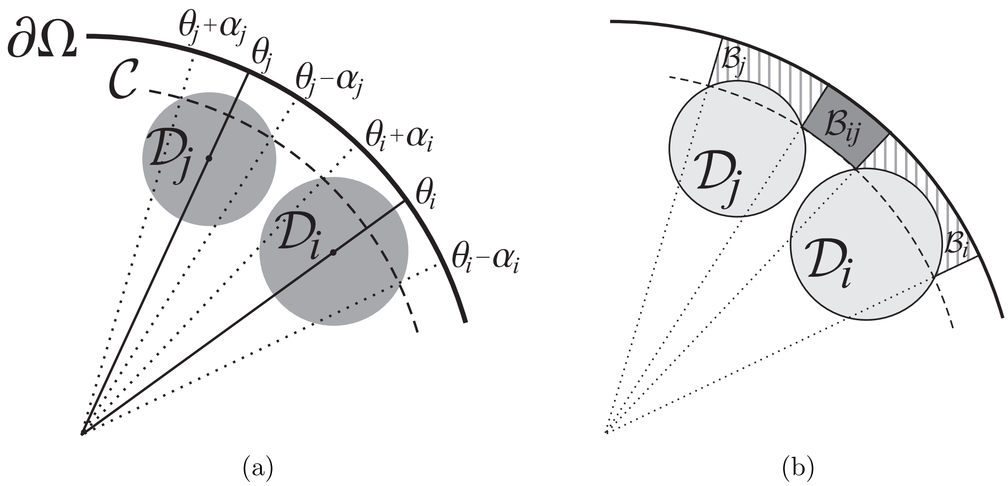


Figure 4.1: (a) The angles of the partition. (b) The partition of B_0.

Notice that $d(\theta)$ is the distance between the two boundaries of B_0, it is a positive function of θ. Under this construction, if the layer width $d(\theta) \equiv d$ does not depend on θ,

$$\Delta\phi = (\frac{\partial^2 w_k}{\partial r^2} + \frac{1}{r}\frac{\partial w_k}{\partial r} - \frac{k^2}{r^2} w_k) \cos k\theta + (\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r})\mathcal{L}(\mathcal{U}) = 0$$

because $\mathcal{L}(\mathcal{U})$ is linear on θ. This is the reason we construct the weight functions like (4.9) and (4.11). Later we will see that this is still a good construction when layer width $d(\theta)$ depends on θ.

Construction in Δ_B

The small domains Δ_{ij} are gaps between Π_B and B_0, they have very small areas and the energy in these domains should also be small compared to the total energy. We need to construct the trial functions in these domains as bridges which can connect the trial functions in Π_B and B_0, such that the trial functions belongs to $H^1(B)$ when we combine them together.

As mentioned before, we need to construct trial functions in different domains such that they match each others in L^2 sense, see Lemma A.4.1. Notice that on the boundary $\partial\Pi_{ij}^- = \partial\Pi_{ij} \cap \partial\Delta_{ij}$, the trial function ϕ is linear,

$$\phi|_{\partial\Pi_{ij}^-} = \frac{1}{2}(\mathcal{U}_i + \mathcal{U}_j) + \frac{y}{h_{ij}(-S_{ij}^-)}(\mathcal{U}_i - \mathcal{U}_j). \tag{4.15}$$

We can put the neck Π_{ij} into the domain D, and transfer the coordinate system from (x, y) to (r, θ). See Figure 2.6(b). We can define the following parameter along $\partial\Pi_{ij}^-$

$$\ell_{ij}^-(\theta) = \frac{1}{2} - \frac{y(r,\theta)}{h_{ij}(-S_{ij}^-)} \in [0, 1] \tag{4.16}$$

Then (4.15) becomes

$$\phi|_{\partial\Pi_{ij}^-} = \left(1 - \ell_{ij}^-(\theta)\right)\mathcal{U}_i + \ell_{ij}^-(\theta)\mathcal{U}_j \tag{4.17}$$

In order to ensure the whole trial function belongs to $H^1(B)$, we construct the trial functions ϕ in Δ_{ij} such that it satisfies the boundary condition (4.17) and

$$\phi|_{\partial\Delta_{ij}\cap\partial B_0} = (1 - \ell_{ij}(\theta))\,\mathcal{U}_i + \ell_{ij}(\theta)\mathcal{U}_j \tag{4.18}$$

where $\ell_{ij}(\theta)$ is defined in (4.14). From the lemma A.4.1, we see that the function ϕ

belongs to $H^1(B)$, because we let the trial functions match each other on all interfaces between different domains.

Also from the Kirszbraun's theorem Lemma 3.2.2 and the results in Section B.2, we have

$$\frac{1}{2}\int_{\Delta_{ij}} |\nabla\phi|^2 \leq C(\mathcal{U}_i - \mathcal{U}_j)^2 = \frac{1}{2}g_{ij}(\mathcal{U}_i - \mathcal{U}_j)^2[O(\sqrt{\frac{\delta_{ij}}{R}})]. \tag{4.19}$$

Hence we have the following approximation for upper bound in $\Pi_B \cup \Delta_B$

$$\frac{1}{2}\int_{\Pi_B\cup\Delta_B} |\nabla\phi|^2 = \sum_{\Pi_{ij}\subset B} \frac{1}{2}g_{ij}(\mathcal{U}_i - \mathcal{U}_j)^2[1 + O(\sqrt{\frac{\delta}{R}})] \tag{4.20}$$

Also, the trial function ϕ satisfies all the constraints in the space V_B. Which means ϕ is a qualified trial function to give $\mathcal{E}_B(\cos k\theta, \mathcal{U})$ an upper bound. And we have

$$\frac{1}{2}\int_{B} |\nabla\phi|^2 = \sum_{\Pi_{ij}\subset B} \frac{1}{2}g_{ij}(\mathcal{U}_i - \mathcal{U}_j)^2[1 + O(\sqrt{\frac{\delta}{R}})] + \frac{1}{2}\int_{B_0} |\nabla\phi|^2 \tag{4.21}$$

where ϕ is the trial function constructed in this section.

4.1.2 The lower bound for $\mathcal{E}_B(\cos k\theta, \mathcal{U})$

Next we will construct the trial functions in B for the lower bound of $\mathcal{E}_B(\cos k\theta, \mathcal{U})$. From (4.4) and (4.5), we need to construct a vector function $\mathbf{j} \in W_B$ which would give $\mathcal{E}_B(\cos k\theta, \mathcal{U})$ an lower bound. We are going to construct $\mathbf{j} \in W_B$ from parts by parts just like what we did for the upper bound.

Our approximation method is going to construct trial functions for the upper and lower bounds, and then using either the upper or the lower bound as an approximation for the energy. This motivate us to think about the gap between the upper and lower bounds before we are constructing trial functions.

To analyze the gap between the upper and low bounds, also to get some clue to construct the trial functions for the lower bound, we need the following lemma

Lemma 4.1.1. *Suppose* $\phi \in V_B$ *gives* $\mathcal{E}_B(\psi,\mathcal{U})$ *an upper bound* $\overline{\mathcal{E}_B(\psi,\mathcal{U})}$ *and* $\mathbf{j} \in W_B$ *gives* $\mathcal{E}_B(\psi,\mathcal{U})$ *a lower bound* $\underline{\mathcal{E}_B(\psi,\mathcal{U})}$, *then the gap between the upper and lower bounds is*

$$G(\phi,\mathbf{j}) = \overline{\mathcal{E}_B(\psi,\mathcal{U})} - \underline{\mathcal{E}_B(\psi,\mathcal{U})} = \frac{1}{2}\int_B |\nabla\phi - \mathbf{j}|^2. \tag{4.22}$$

Proof. By Green's identity

$$\begin{aligned}
\int_B \nabla\phi\cdot\mathbf{j} &= \int_B \nabla\phi\cdot\mathbf{j} + \int_B \phi\nabla\cdot\mathbf{j} \\
&= \int_{\partial D} \psi\mathbf{j}\cdot\mathbf{n} + \sum_{i\in\mathcal{S}_B} \mathcal{U}_i \int_{\partial D_i\cap\partial B} \mathbf{j}\cdot\mathbf{n}
\end{aligned}$$

Then we have

$$\begin{aligned}
G(\phi,\mathbf{j}) &= \frac{1}{2}\int_B |\nabla\phi|^2 + \frac{1}{2}\int_B |\mathbf{j}|^2 - \int_{\partial D} \psi\mathbf{j}\cdot\mathbf{n} - \sum_{i\in\mathcal{S}_B} \mathcal{U}_i \int_{\partial D_i\cap\partial B} \mathbf{j}\cdot\mathbf{n} \\
&= \frac{1}{2}\int_B |\nabla\phi|^2 + \frac{1}{2}\int_B |\mathbf{j}|^2 - \int_B \nabla\phi\cdot\mathbf{j} \\
&= \frac{1}{2}\int_B |\nabla\phi - \mathbf{j}|^2
\end{aligned}$$

□

The lemma (4.1.1) suggests us to construct the flux $\mathbf{j}$ as close to $\nabla\phi$ as possible to make the gap (4.22) small. Notice that the flux $\mathbf{j} \in W_B$ is only required to belong to the space $\mathbf{L}^2(B)$, so we can construct the flux $\mathbf{j}$ in Π_B, Δ_B and B_0 separately.

Construction in Π_B

In the necks Π_{ij}, we will still use the same coordinate system when we construct ϕ for the upper bound. As mentioned before, the flux will be strong in the horizontal direction but weak in vertical direction. We construct $\mathbf{j}$ in Π_{ij} like

$$\mathbf{j} = \left(0, \frac{\mathcal{U}_i - \mathcal{U}_j}{h_{ij}(x)}\right)^T, \tag{4.23}$$

here we use the same coordinate system as in the construction (4.6). It is easy to check that $\mathbf{j}$ is divergence free in Π_{ij} and $\mathbf{j} \cdot \mathbf{n} = 0$ on $\partial\Pi_{ij} \cap \partial\Delta$. Also we have the gap in Π_{ij}

$$G_{\Pi_{ij}} := \frac{1}{2}\int_{\Pi_{ij}} |\nabla\phi - \mathbf{j}|^2 = [O(\sqrt{\frac{\delta}{R}})] \int_{\Pi_{ij}} |\nabla\phi|^2 \tag{4.24}$$

from the approximation (4.7) and the discusion in Section 2.3.

Construction in Δ_B

In the small areas Δ_{ij}, we just let

$$\mathbf{j} = (0,0)^T. \tag{4.25}$$

Then we have the gap in Δ_{ij}

$$G_{\Delta_{ij}} := \frac{1}{2}\int_{\Delta_{ij}} |\nabla\phi - \mathbf{j}|^2 = \frac{1}{2}\int_{\Delta_{ij}} |\nabla\phi|^2 = [O(\sqrt{\frac{\delta}{R}})] \int_{\Pi_{ij}} |\nabla\phi|^2 \tag{4.26}$$

from the approximation (4.19)

Construction in B_0

In the layer B_0, we will use the polar coordinate system as in the formula (4.8), also see appendix A.5 for more details of the polar coordinate system. First we construct a trial function $H(r,\theta) \in H^1(B_0)$, then we let

$$\mathbf{j} = \nabla^\perp H = -\frac{1}{r}\frac{\partial H}{\partial \theta}\mathbf{u}_r + \frac{\partial H}{\partial r}\mathbf{u}_\theta.$$

In this way, $\mathbf{j}$ will be divergence free in B_0 and there is no other requirements for $\mathbf{j}$ in B_0,

In order to make the gap in (4.22) small, we need to choose $H(r,\theta)$ such that $|\nabla\phi - \nabla^\perp H|$ is small. In general we cannot choose $H(r,\theta)$ such that $\nabla^\perp H = \nabla\phi$, otherwise $\Delta\phi = \nabla\cdot(\nabla\phi) = \nabla\cdot(\nabla^\perp H) = 0$. But this is not true for a general trial function in V_B, at least it is not true for ϕ in (4.8). However, we can choose $H(r,\theta)$ such that $\nabla\phi$ and $\nabla^\perp H$ equals to each other in one direction. We let

$$H(r,\theta) = F(\theta) - \int_r^L \frac{1}{s}\frac{\partial\phi(s,\theta)}{\partial\theta}ds \tag{4.27}$$

where ϕ is the function showed in (4.8) and $F(\theta)$ is a function need to be determined. By this construction

$$\frac{\partial H}{\partial r} = \frac{1}{r}\frac{\partial\phi}{\partial\theta}. \tag{4.28}$$

In conclusion, the gap in B_0 is

$$\begin{aligned} G_{B_0} &:= \frac{1}{2}\int_{B_0}|\nabla\phi - \mathbf{j}|^2 = \frac{1}{2}\int_{B_0}|\nabla\phi - \nabla^\perp H|^2 \\ &= \frac{1}{2}\int_{B_0}\left\{\frac{\partial\phi}{\partial r} + \frac{1}{r}F'(\theta) - \frac{1}{r}\int_r^L \frac{1}{s}\frac{\partial^2\phi(s,\theta)}{\partial\theta^2}ds\right\}^2 \end{aligned} \tag{4.29}$$

Choose $F(\theta)$ such that

$$F'(\theta) = -\frac{\partial \phi}{\partial r}(L, \theta) \tag{4.30}$$

By this assumption, $\mathbf{j}|_{r=L} = \nabla\phi|_{r=L}$, which means $\mathbf{j}$ and $\nabla\phi$ are totally matched at the boundary ∂D. Then we will have

$$\begin{aligned}
G_{B_0} &= \frac{1}{2}\int_{B_0}\left\{-\frac{1}{r}\int_r^L \frac{\partial}{\partial s}(s\frac{\partial \phi}{\partial s}) - \frac{1}{r}\int_r^L \frac{1}{s}\frac{\partial^2\phi(s,\theta)}{\partial\theta^2} ds\right\}^2 \\
&= \frac{1}{2}\int_{B_0}\left\{\frac{1}{r}\int_r^L s\frac{\partial^2\phi(s,\theta)}{\partial s^2} + \frac{\partial\phi(s,\theta)}{\partial s} + \frac{1}{s}\frac{\partial^2\phi(s,\theta)}{\partial\theta^2} ds\right\}^2 \\
&= \frac{1}{2}\int_{B_0}\left\{\frac{1}{r}\int_r^L s\Delta\phi(s,\theta)\right\}^2
\end{aligned}$$

4.1.3 Bounds of the gap

From (4.24) and (4.26), we will have

$$\begin{aligned}
G_{\Pi_B\cup\Delta_B} &= \sum_{\Pi_{ij}\subset B}(G_{\Pi_{ij}} + G_{\Delta_{ij}}) = O(\sqrt{\frac{\delta}{R}})\sum_{\Pi_{ij}\subset B}\frac{1}{2}\int_{\Pi_{ij}\cup\Delta_{ij}}|\nabla\phi|^2 \\
&= \frac{1}{2}O(\sqrt{\frac{\delta}{R}})\sum_{\Pi_{ij}\subset B} g_{ij}(\mathcal{U}_i - \mathcal{U}_j)^2.
\end{aligned} \tag{4.31}$$

Then we are going to prove

$$G_{B_0} = \frac{1}{2}\int_{B_0}\left\{\frac{1}{r}\int_r^L s\Delta\phi(s,\theta)\right\}^2 = O(1) = \frac{1}{2}O(\sqrt{\frac{\delta}{R}})\int_{B_0}|\nabla\phi|^2 \tag{4.32}$$

for any k and $\mathcal{U}$.

In the domains B_{ij}, the width of the layer $d(\theta) = R/2$ is a constant, neither w_k nor w depends on θ. And

$$\mathcal{L}(\mathcal{U}) = (1 - \ell_{ij}(\theta))\mathcal{U}_i + \ell_{ij}(\theta)\mathcal{U}_j \tag{4.33}$$

with ℓ_{ij} defined in (4.14).

So in this area

$$\begin{aligned}\frac{\partial \mathcal{L}(\mathcal{U})}{\partial \theta} &= \frac{\mathcal{U}_j - \mathcal{U}_i}{(\theta_j - \alpha_j) - (\theta_i + \alpha_i)} \\ \frac{\partial^2 \mathcal{L}(\mathcal{U})}{\partial \theta^2} &= 0\end{aligned} \tag{4.34}$$

From the construction (4.8), we have

$$\begin{aligned}\frac{\partial \phi}{\partial r} &= \frac{\partial w_k}{\partial r}\cos k\theta + \frac{\partial w}{\partial r}\mathcal{L}(\mathcal{U}) \\ \frac{\partial^2 \phi}{\partial r^2} &= \frac{\partial^2 w_k}{\partial r^2}\cos k\theta + \frac{\partial^2 w}{\partial r^2}\mathcal{L}(\mathcal{U}) \\ \frac{\partial \phi}{\partial \theta} &= -k w_k \sin k\theta + w\frac{\partial \mathcal{L}(\mathcal{U})}{\partial \theta} \\ \frac{\partial^2 \phi}{\partial \theta^2} &= -k^2 w_k \cos k\theta\end{aligned} \tag{4.35}$$

So we will have

$$\begin{aligned}\Delta\phi &= \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} \\ &= (\frac{\partial^2 w_k}{\partial r^2} + \frac{1}{r}\frac{\partial w_k}{\partial r} - \frac{k^2}{r^2}w_k)\cos k\theta + (\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r})\mathcal{L}(\mathcal{U}) \\ &= 0\end{aligned} \tag{4.36}$$

which means the gap

$$G_{B_{ij}} = 0$$

in the domain B_{ij}.

In the domains B_i, $\mathcal{L}(\mathcal{U}) = \mathcal{U}_i$ is a constant. From the construction (4.8), we have

$$\begin{aligned}
\frac{\partial\phi}{\partial r} &= \frac{\partial w_k}{\partial r}\cos k\theta + \frac{\partial w}{\partial r}\mathcal{U}_i \\
\frac{\partial^2\phi}{\partial r^2} &= \frac{\partial^2 w_k}{\partial r^2}\cos k\theta + \frac{\partial^2 w}{\partial r^2}\mathcal{U}_i \\
\frac{\partial\phi}{\partial\theta} &= -kw_k\sin k\theta + \frac{\partial w_k}{\partial\theta}\cos k\theta + \frac{\partial w}{\partial\theta}\mathcal{U}_i \\
\frac{\partial^2\phi}{\partial\theta^2} &= -k^2 w_k\cos k\theta - 2k\frac{\partial w_k}{\partial\theta}\sin k\theta + \frac{\partial^2 w_k}{\partial\theta^2}\cos k\theta + \frac{\partial^2 w}{\partial\theta^2}\mathcal{U}_i
\end{aligned} \tag{4.37}$$

So we will have

$$\begin{aligned}
\Delta\phi &= \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \\
&= (\frac{\partial^2 w_k}{\partial r^2} + \frac{1}{r}\frac{\partial w_k}{\partial r} - \frac{k^2}{r^2}w_k)\cos k\theta + (\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r})\mathcal{U}_i \\
&+ \frac{1}{r^2}\left\{-2k\frac{\partial w_k}{\partial\theta}\sin k\theta + \frac{\partial^2 w_k}{\partial\theta^2}\cos k\theta + \frac{\partial^2 w}{\partial\theta^2}\mathcal{U}_i\right\} \\
&= \frac{1}{r^2}\left\{-2k\frac{\partial w_k}{\partial\theta}\sin k\theta + \frac{\partial^2 w_k}{\partial\theta^2}\cos k\theta + \frac{\partial^2 w}{\partial\theta^2}\mathcal{U}_i\right\}
\end{aligned} \tag{4.38}$$

Hence the gap in B_i is

$$G_{B_i} = \frac{1}{2}\int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} G_i(\theta)d\theta \tag{4.39}$$

with

$$\begin{aligned}
&G_i(\theta) = \int_{L-d(\theta)}^{L} rdr\left\{\frac{1}{r}\int_r^L s\Delta\phi(s,\theta)ds\right\}^2 \\
&= \int_{L-d(\theta)}^{L}\frac{dr}{r}\left\{\int_r^L\frac{ds}{s}\left(-2k\frac{\partial w_k(s,\theta)}{\partial\theta}\sin k\theta + \frac{\partial^2 w_k(s,\theta)}{\partial\theta^2}\cos k\theta + \frac{\partial^2 w(s,\theta)}{\partial\theta^2}\mathcal{U}_i\right)\right\}^2 \\
&\leq 3G_{i1}^k(\theta) + 3G_{i2}^k(\theta) + 3\mathcal{U}_i^2 G_{i3}(\theta)
\end{aligned} \tag{4.40}$$

where

$$
\begin{aligned}
G_{i1}^{k}(\theta) &= \int_{L-d(\theta)}^{L} \frac{dr}{r}\left(\int_{r}^{L}\frac{ds}{s}(-2k\frac{\partial w_k(s,\theta)}{\partial\theta})\right)^2 \\
G_{i2}^{k}(\theta) &= \int_{L-d(\theta)}^{L} \frac{dr}{r}\left(\int_{r}^{L}\frac{ds}{s}\frac{\partial^2 w_k(s,\theta)}{\partial\theta^2}\right)^2 \\
G_{i3}(\theta) &= \int_{L-d(\theta)}^{L} \frac{dr}{r}\left(\int_{r}^{L}\frac{ds}{s}\frac{\partial^2 w(s,\theta)}{\partial\theta^2}\right)^2
\end{aligned}
\tag{4.41}
$$

Here we used Cauchy-Schwars inequality.

In the appendix B, we will prove that

$$
G_{B_i} = \frac{1}{2}\int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} G_i(\theta) = O(1) = \frac{1}{2}O(\sqrt{\frac{\delta}{R}})\int_{B_i}|\nabla\phi|^2 \tag{4.42}
$$

which will not blow up for any k as $\delta \to 0$.

Then the gap in B_0 is

$$
G_{B_0} = \sum_{i=1}^{N_B} G_{B_i} = \sum_{i=1}^{N_B}\frac{1}{2}\int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} G_i(\theta)d\theta = \frac{1}{2}O(\sqrt{\frac{\delta}{R}})\int_{B_0}|\nabla\phi|^2 \tag{4.43}
$$

In the summary, by the construction of ϕ for upper bound and the related construction $\mathbf{j}$ for the lower bound, we have the following gap in B between the upper and lower bounds for the energy in B with boundary condition $\cos(k\theta)$ and any vector $\mathcal{U}$

$$
G(\phi,\mathbf{j}) = \frac{1}{2}O(\sqrt{\frac{\delta}{R}})\int_{B}|\nabla\phi|^2
$$

It means

$$
\mathcal{E}_B(\cos k\theta,\mathcal{U}) = \left(\frac{1}{2}\sum_{\Pi_{ij}\subset B} g_{ij}(\mathcal{U}_i-\mathcal{U}_j)^2 + \frac{1}{2}\int_{B_0}|\nabla\phi|^2\right)[1+O(\sqrt{\frac{\delta}{R}})] \tag{4.44}
$$

where ϕ in B_0 is defined in (4.8).

4.2 The approximation for $\mathcal{E}_B(\cos k\theta, \mathcal{U})$

Now we use the formula in (4.44) to approximate $\mathcal{E}_B(\cos k\theta, \mathcal{U})$. We need to approximate the following integral in the B_0

$$\int_{B_0} |\nabla\phi|^2 = \sum_{i=1}^{N_B} \int_{B_i} |\nabla\phi|^2 + \sum_{B_{ij} \subset B_0} \int_{B_{ij}} |\nabla\phi|^2$$

with ϕ given in the equation (4.8).

4.2.1 The approximation in B_{ij}

Notice that B_{ij} is the area between the neighbor disk D_i, D_j and the boundary ∂D, see Figure 4.1(a) and Figure 4.1(b). Suppose the center of D_i, D_j are located at (r_i, θ_i) and (r_j, θ_j) respectively. We suppose that $\theta_i < \theta_j$ and denote the angle

$$\alpha_{ij} := \frac{1}{2}\left((\theta_j - \alpha_j) - (\theta_i + \alpha_i)\right) \tag{4.45}$$

It will satisfy $\alpha_{ij} = O(R/L)$ in our construction when we suppose $\delta \ll R$.

We need to approximate

$$\int_{B_{ij}} |\nabla\phi|^2 = \int_{\theta_i+\alpha_i}^{\theta_j-\alpha_j} d\theta \int_{L-R/2}^{L} r dr |\nabla\phi|^2$$

Here the layer width $d(\theta) \equiv R/2$, and the weight function w_k, w will only depend on r but not on θ.

From the construction for ϕ in (4.8), we will have

$$|\nabla\phi|^2 = \left(\frac{\partial w_k}{\partial r}\cos k\theta + \frac{\partial w}{\partial r}\mathcal{L}(\mathcal{U})\right)^2 + \frac{1}{r^2}\left(-k w_k \sin k\theta + w\frac{\mathcal{U}_j - \mathcal{U}_i}{2\alpha_{ij}}\right)^2$$

Integrate this in B_{ij} will give us

$$
\begin{aligned}
\int_{B_{ij}} |\nabla\phi|^2 &= k\alpha_{ij} + \frac{2kp^2}{1-p^2}\alpha_{ij} - \frac{2k^2p^2\ln(1-R/(2L))}{(1-p^2)^2}\int_{\theta_i+\alpha_i}^{\theta_j-\alpha_j}\cos 2k\theta d\theta \\
&+ \frac{2}{\ln(1-R/(2L))}\int_{\theta_i+\alpha_i}^{\theta_j-\alpha_j}\mathcal{L}(\mathcal{U})\cos k\theta d\theta - \frac{1}{\ln(1-R/(2L))}\int_{\theta_i+\alpha_i}^{\theta_j-\alpha_j}(\mathcal{L}(\mathcal{U}))^2 d\theta \quad (4.46) \\
&+ 2(\mathcal{U}_j-\mathcal{U}_i)\frac{1-p^2+2p\ln p}{\ln p(1-p^2)} + (\mathcal{U}_j-\mathcal{U}_i)^2\frac{-\ln(1-R/(2L))}{6\alpha_{ij}}
\end{aligned}
$$

where $p = (1-R/(2L))^k$.

From the proposition B.1.1, we have

$$
\frac{2kp^2}{1-p^2}\alpha_{ij} = \frac{2k(1-R/(2L))^{2k}}{1-(1-R/(2L))^{2k}}\alpha_{ij} \leq \frac{2(1-R/(2L))^2}{1-(1-R/(2L))^2}\alpha_{ij} = O(1),
$$

Because $|\cos k\theta| \leq 1, |\mathcal{U}_i| \leq 1, |\mathcal{U}_j| \leq 1, |L(\mathcal{U})| \leq 1$ and

$$
\frac{1}{\ln(1-R/(2L))} = -\frac{2L}{R} + O(1)
$$

we can have

$$
\begin{aligned}
\left|\frac{2}{\ln(1-R/(2L))}\int_{\theta_i+\alpha_i}^{\theta_j-\alpha_j}\mathcal{L}(\mathcal{U})\cos k\theta d\theta\right| &\leq C\frac{L\alpha_{ij}}{R} = O(1) \\
\left|-\frac{1}{\ln(1-R/(2L))}\int_{\theta_i+\alpha_i}^{\theta_j-\alpha_j}(\mathcal{L}(\mathcal{U}))^2 d\theta\right| &\leq C\frac{L\alpha_{ij}}{R} = O(1) \\
\left|(\mathcal{U}_j-\mathcal{U}_i)^2\frac{-\ln(1-R/(2L))}{3\alpha_{ij}}\right| &\leq C\frac{R}{2L\alpha_{ij}} = O(1)
\end{aligned}
$$

We can prove

$$
\left|\frac{1-p^2+2p\ln p}{\ln p(1-p^2)}\right| \leq 1, \qquad \text{for all } p \in (0,1)
$$

Similarly like the proof in appendix B.4, we can prove for some constant C

$$\left|\frac{2k^2p^2\ln(1-R/(2L))}{(1-p^2)^2}\right| \leq \frac{CL}{R}, \qquad \text{for all } k$$

So we will have

$$\left|\frac{2k^2p^2\ln(1-R/(2L))}{(1-p^2)^2}\int_{\theta_i+\alpha_i}^{\theta_j-\alpha_j}\cos 2k\theta d\theta\right| \leq \frac{CL}{R}\alpha_{ij} = O(1)$$

In the summary we will have

$$\int_{B_{ij}}|\nabla\phi|^2 = k\alpha_{ij} + O(1). \tag{4.47}$$

Notice that when k is small,$k\alpha_{ij}$ will also be $O(1)$. However when k grows, $k\alpha_{ij}$ will blow up.

4.2.2 The approximation in B_i

Remember the formular (2.28) in Section 2.3, we define the effective conductance of the boundary neck, which is B_i here, as

$$g_i = \pi\sqrt{\frac{2LR_i}{(L-R_i)\delta_i}}.$$

It is actually an approximation of

$$\pi\sqrt{\frac{2LR_i}{r_i\delta_i}} = \pi\sqrt{\frac{2LR_i}{(L-R_i)\delta_i}} + O(1).$$

This is because

$$r_i + R_i + \delta_i = L \text{ and } \delta_i \ll R_i \ll L.$$

We will show how can we get the formula for effective conductance g_i in the approximation for integral in B_i.

Now let's approximate the integral in B_i, which is the area between the disk D_i and the boundary ∂D, see Figure 4.1(b)

$$\int_{B_i} |\nabla\phi|^2 = \int_{\theta_i-\alpha_i}^{\theta_i+\alpha_j} d\theta \int_{L-d(\theta)}^{L} rdr|\nabla\phi|^2$$

where $d(\theta)$ is a function of θ defined in (2.45).

In B_i, we have

$$|\nabla\phi|^2 = \left(\frac{\partial w_k}{\partial r}\cos k\theta + \frac{\partial w}{\partial r}\mathcal{U}_i\right)^2 + \frac{1}{r^2}\left(-kw_k \sin k\theta + \frac{\partial w_k}{\partial\theta}\cos k\theta + \frac{\partial w}{\partial\theta}\mathcal{U}_i\right)^2$$

and we want to write the integral into a quadratic form of $\mathcal{U}_i$ like following

$$\int_{B_i} |\nabla\phi|^2 := a_i\mathcal{U}_i^2 + 2b_i\mathcal{U}_i + c_i \tag{4.48}$$

After some calculation, we will have

$$\begin{aligned}
a_i &= \int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} d\theta \int_{L-d(\theta)}^{L} rdr \left\{(\frac{\partial w}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial w}{\partial\theta})^2\right\} \\
b_i &= \int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} d\theta \int_{L-d(\theta)}^{L} rdr \left\{(\frac{\partial w_k}{\partial r}\cos k\theta)\frac{\partial w}{\partial r} + \frac{1}{r^2}(-kw_k\sin k\theta + \frac{\partial w_k}{\partial\theta}\cos k\theta)\frac{\partial w}{\partial\theta}\right\} \\
c_i &= \int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} d\theta \int_{L-d(\theta)}^{L} rdr \left\{\left(\frac{\partial w_k}{\partial r}\cos k\theta\right)^2 + \frac{1}{r^2}\left(-kw_k\sin k\theta + \frac{\partial w_k}{\partial\theta}\cos k\theta\right)^2\right\}
\end{aligned}$$

By the approximation of a_i, b_i and c_i in Section B.5, we will show that

$$\begin{aligned}
&\int_{B_i} |\nabla\phi|^2 = a_i\mathcal{U}_i^2 + 2b_i\mathcal{U}_i + c_i \\
&= \pi\sqrt{\frac{2LR}{r_i\delta_i}}\left(\mathcal{U}_i - \exp\left[-k\sqrt{\frac{2R\delta_i}{Lr_i}}\right]\cos k\theta_i\right)^2 + k\alpha_i + \mathcal{R}_{ki} + O(1) \qquad (4.49)\\
&= g_i\left(\mathcal{U}_i - \exp\left[-\frac{k\delta_i}{L}g_i\right]\cos k\theta_i\right)^2 + k\alpha_i + \mathcal{R}_{ki} + O(1)
\end{aligned}$$

where the resonance term $\mathcal{R}_{ki}$ is

$$\begin{aligned}
\mathcal{R}_{ki} &= \frac{\pi}{2}\sqrt{\frac{2LR}{r_i\delta_i}}\left\{\sqrt{\frac{2k\delta_i}{L\pi}}Li_{1/2}\left(\exp\left[-\frac{2k\delta_i}{L}\right]\right) - \exp\left[-2k\sqrt{\frac{2R\delta_i}{Lr_i}}\right]\right\} \\
&= \frac{g_i}{2}\left\{\sqrt{\frac{2k\delta_i}{L\pi}}Li_{1/2}\left(\exp\left[-\frac{2k\delta_i}{L}\right]\right) - \exp\left[-\frac{2k\delta_i}{L}g_i\right]\right\}
\end{aligned} \qquad (4.50)$$

Here Li is the Polylogarithm function which is defined by

$$Li_s(z) := \sum_{k=1}^{\infty}\frac{z^k}{k^s}. \qquad (4.51)$$

and it has the following asymptotical expansion

$$Li_{1/2}(e^{-x}) = \sqrt{\frac{\pi}{x}} + \zeta(1/2) - \zeta(-1/2)x + O(x^{3/2}) \text{ for } 0 < x \ll 1, \qquad (4.52)$$

where ζ is the Riemann zeta function.

Remark 4.2.1. *In the results* (4.49)*, the first term of the second line is the network effect term. It is similar to the energy in the necks we discussed before, and it has the same singularity order. The* $k\alpha_i$ *term represents the energy for the tangential fluxes. The term* $\mathcal{R}_{ki}$ *is the resonance term, and we will discuss it carefully in this section. The term shows up in the results also because of the inclusions, however it is more* *complicated than the network effect.*

4.2.3 Asymptotic approximation of the resonance

In this section, we will discuss the asymptotic approximation for the resonance term in (4.50). We will see how does the resonance term affect the energy in each B_i as k grows.

We will first introduce the small parameter ϵ_i for convenience

$$\epsilon_i := \frac{\delta_i}{R} \ll 1,$$

However, in the results or other places of this thesis, we will not use this parameter.

Notice that, we have the following approximation

$$\sqrt{\frac{2L}{r_i}} = \sqrt{\frac{2L}{L-R-\delta_i}} = \sqrt{\frac{2L}{L-R}} + O(\epsilon_i). \tag{4.53}$$

In the discuss of this section, we introduce a new parameter here

$$\mu_i := -\frac{\ln(2kR/L)}{\ln \epsilon_i}, \tag{4.54}$$

which will give us

$$\frac{2kR}{L} = \epsilon_i^{-\mu_i}.$$

Notice that μ_i grows as k grows. The introduction of this parameter build connections between k and δ_i/R, which is an important scale in our approximation.

Then we have

$$k\alpha_i = O(\epsilon^{-\mu_i}).$$

because $\alpha_i = O(R/L)$. This formula only gives the order of $k\alpha_i$, but we will keep the expression $k\alpha_i$ in our formulas.

There will be several different cases for discussion of the asymptotic properties of the resonance term.

1. When $\mu_i \le 0$, which means k is small such that $kR/L \le 1$. By (4.52) and the Taylor expansion of e^x for small x, we will easily have

$$\begin{aligned}
\mathcal{R}_{ki} &= \frac{\pi}{2}\sqrt{\frac{2L}{r_i}}\epsilon_i^{-1/2}\left\{\frac{\epsilon_i^{\frac{1-\mu_i}{2}}}{\sqrt{\pi}}Li_{1/2}\left(\exp\left[-\epsilon_i^{1-\mu_i}\right]\right) - \exp\left[-\sqrt{\frac{2L}{r_i}}\epsilon_i^{1/2-\mu_i}\right]\right\} \\
&= \frac{\pi}{2}\sqrt{\frac{2L}{r_i}}\epsilon_i^{-1/2}\left\{1 + O(\epsilon_i^{(1-\mu_i)/2}) - 1 + O(\epsilon_i^{1/2-\mu_i})\right\} \\
&= O(\epsilon_i^{-\mu_i/2}) + O(\epsilon_i^{-\mu_i}) \le O(1).
\end{aligned} \tag{4.55}$$

What is more, the integral in (4.49) will be simplified to

$$\int_{B_i} |\nabla\phi|^2 = g_i\left(\mathcal{U}_i - \cos k\theta_i\right)^2 + O(1). \tag{4.56}$$

This matches the results in [7] and [4, 5]. In their papers, they discussed the problems with piecewise constant boundary condition or boundary conditions without highly oscillation. They use one point potential to approximate the boundary condition near the inclusion D_i, and use a neck with some effect conductance to simulate the part between ∂D_i and the domain boundary ∂D.

In this case the mainly contribution of the energy comes from the neck effect, and it is $O(\epsilon_i^{-1/2})$. The contribution from the tangential flow $k\alpha_i$ is $O(1)$, which is hidden in approximation error.

2. When $0 < \mu_i < 1/2$, the contribution from the tangential flow $k\alpha_i$ is $O(\epsilon_i^{-\mu_i}) \gg 1$ now. In this case we can still use (4.52) and the Taylor expansion of e^x for small x, we will have

$$
\begin{aligned}
\mathcal{R}_{ki} &= \frac{\pi}{2}\sqrt{\frac{2L}{r_i}}\epsilon_i^{-1/2}\left\{\frac{\epsilon_i^{\frac{1-\mu_i}{2}}}{\sqrt{\pi}}Li_{1/2}\left(\exp\left[-\epsilon_i^{1-\mu_i}\right]\right) - \exp\left[-\sqrt{\frac{2L}{r_i}}\epsilon_i^{1/2-\mu_i}\right]\right\} \\
&= \frac{\pi}{2}\sqrt{\frac{2L}{r_i}}\epsilon_i^{-1/2}\left\{1 + O(\epsilon_i^{(1-\mu_i)/2}) - 1 + \sqrt{\frac{2L}{r_i}}\epsilon_i^{1/2-\mu_i} - O(\epsilon_i^{2(1/2-\mu_i)})\right\} \\
&= \frac{\pi}{2}\frac{2L}{r_i}\epsilon_i^{-\mu_i} + O(\epsilon_i^{-\mu_i/2}) + O(\epsilon_i^{-\mu_i+(1/2-\mu_i)}) \\
&= \frac{\pi}{2}\frac{2L}{r_i}\frac{2kR}{L} + O(\epsilon_i^{-\mu_i/2}) + O(\epsilon_i^{-\mu_i+(1/2-\mu_i)}) \\
&= \frac{\pi}{2}\frac{2L}{L-R}\frac{2kR}{L} + O(\epsilon_i^{-\mu_i/2}) + O(\epsilon_i^{-\mu_i+(1/2-\mu_i)}) \\
&= 2k\pi\frac{R}{L-R} + O(\epsilon_i^{-\mu_i/2}) + O(\epsilon_i^{-\mu_i+(1/2-\mu_i)})
\end{aligned}
\tag{4.57}
$$

In this case, we cannot write the integral in (4.49) into the form like

$$
\int_{B_i} |\nabla\phi|^2 = O(\epsilon_i^{-1/2}) + O(1).
$$

However, we know the first two leading order terms of the approximation for the integral

$$
\begin{aligned}
\int_{B_i} |\nabla\phi|^2 = k\alpha_i &+ g_i\left(\mathcal{U}_i - \exp\left[-k\sqrt{\frac{2R\delta_i}{L(L-R)}}\right]\cos k\theta_i\right)^2 \\
&+ 2k\pi\frac{R}{L-R} + O(\epsilon_i^{-\mu_i/2}) + O(\epsilon_i^{-\mu_i+2(1/2-\mu_i)}) \\
&= k\alpha_i + O(\epsilon_i^{-1/2})[1 + o(1)].
\end{aligned}
\tag{4.58}
$$

3. When $\mu_i = 1/2$, which is a special case. We can still use (4.52) to get

$$
\begin{aligned}
\mathcal{R}_{ki} &= \frac{\pi}{2}\sqrt{\frac{2L}{r_i}}\epsilon_i^{-1/2}\left\{\frac{\epsilon_i^{\frac{1-\mu_i}{2}}}{\sqrt{\pi}}Li_{1/2}\left(\exp\left[-\epsilon_i^{1-\mu_i}\right]\right) - \exp\left[-\sqrt{\frac{2L}{r_i}}\epsilon_i^{1/2-\mu_i}\right]\right\} \\
&= \frac{\pi}{2}\sqrt{\frac{2L}{r_i}}\epsilon_i^{-1/2}\left\{1 + O(\epsilon_i^{(1-\mu_i)/2}) - \exp[-\sqrt{\frac{2L}{r_i}}]\right\} \\
&= \frac{\pi}{2}\left(1 - \exp[-\sqrt{\frac{2L}{r_i}}]\right)\sqrt{\frac{2L}{r_i}}\epsilon_i^{-1/2} + O(\epsilon_i^{-\mu_i/2}) \\
&= \frac{g_i}{2}\left(1 - \exp[-\sqrt{\frac{2L}{r_i}}]\right) + O(\epsilon_i^{-\mu_i/2})
\end{aligned}
\tag{4.59}
$$

In this case, we have

$$
\begin{aligned}
\int_{B_i} |\nabla\phi|^2 &= k\alpha_i + g_i\left(\mathcal{U}_i - \exp\left[-\sqrt{\frac{2L}{L-R}}\right]\cos k\theta_i\right)^2 \\
&\quad + \frac{g_i}{2}\left(1 - \exp\left[-\sqrt{\frac{2L}{L-R}}\right]\right) + O(\epsilon_i^{-\mu_i/2}) \\
&= k\alpha_i + O(\epsilon_i^{-1/2})[1 + o(1)].
\end{aligned}
\tag{4.60}
$$

4. When $1/2 < \mu_i < 1$, we will have a different expression for the resonance term. In this case

$$
\sqrt{\frac{2L}{r_i}}\epsilon_i^{-1/2}\exp\left[-\sqrt{\frac{2L}{r_i}}\epsilon_i^{1/2-\mu_i}\right] \ll 1,
$$

and we can use (4.52) to get

$$\begin{aligned}\mathcal{R}_{ki} &= \frac{\pi}{2}\sqrt{\frac{2L}{r_i}}\epsilon_i^{-1/2}\left\{\frac{\epsilon_i^{\frac{1-\mu_i}{2}}}{\sqrt{\pi}}Li_{1/2}\left(\exp\left[-\epsilon_i^{1-\mu_i}\right]\right) - \exp\left[-\sqrt{\frac{2L}{r_i}}\epsilon_i^{1/2-\mu_i}\right]\right\} \\ &= \frac{\pi}{2}\sqrt{\frac{2L}{r_i}}\epsilon_i^{-1/2}\left\{1 + O(\epsilon_i^{(1-\mu_i)/2})\right\} + O(1) \\ &= \frac{\pi}{2}\sqrt{\frac{2L}{r_i}}\epsilon_i^{-1/2} + O(\epsilon_i^{-\mu_i/2}) \\ &= \frac{g_i}{2}.\end{aligned} \tag{4.61}$$

In this case, we have

$$\begin{aligned}\int_{B_i}|\nabla\phi|^2 &= k\alpha_i + g_i\left(\mathcal{U}_i - \exp\left[-\sqrt{\frac{2L}{L-R}}\right]\cos k\theta_i\right)^2 + \frac{g_i}{2} \\ &= k\alpha_i + O(\epsilon_i^{-1/2})[1+o(1)].\end{aligned} \tag{4.62}$$

5. When $\mu_i = 1$, we have

$$\mathcal{R}_{ki} = \frac{g_i}{2}\frac{Li_{1/2}(e^{-1})}{\sqrt{\pi}} + O(1). \tag{4.63}$$

$$\begin{aligned}\int_{B_i}|\nabla\phi|^2 &= k\alpha_i + g_i\left(\mathcal{U}_i - \exp\left[-\sqrt{\frac{2L}{L-R}}\right]\cos k\theta_i\right)^2 + \frac{g_i}{2}\frac{Li_{1/2}(e^{-1})}{\sqrt{\pi}} + O(1) \\ &= k\alpha_i + O(\epsilon_i^{-1/2})[1+o(1)].\end{aligned} \tag{4.64}$$

6. When $\mu_i > 1$, we have

$$\mathcal{R}_{ki} \ll 1. \tag{4.65}$$

Hence

$$\int_{B_i} |\nabla\phi|^2 = k\alpha_i + g_i \left(\mathcal{U}_i - \exp\left[-\sqrt{\frac{2L}{L-R}} \right] \cos k\theta_i \right)^2 + O(1) \\ = k\alpha_i + O(\epsilon_i^{-1/2})[1 + o(1)] \tag{4.66}$$

Remark 4.2.2. *We call the term* $\mathcal{R}_{ki}$ *in* (4.50) *a resonance term because it is* $O(1)$*, which almost vanishes in* (4.49)*, when* $kR/L \ll 1$ *or* $k\delta_i/L \gg 1$*. And it does not vanish when* k *is in the medium region.*

Remark 4.2.3. *Whatever* k *is, the value of the integral in* (4.49) *is always* $k\alpha_i + O(\epsilon_i^{-1/2})$*. It means that, the energy because of the existence of inclusions is always* $O(\epsilon_i^{-1/2})$*.*

If we just want to get the leading order $O(\epsilon_i^{-1/2})$ in the resonance term, we can define

$$\mathcal{R}_{ki}^0 = \begin{cases} 0, & \mu_i < \frac{1}{2} \text{ or } \mu_i > 1 \\ \frac{1}{2} g_i \left(1 - \exp[-\sqrt{\frac{2L}{L-R}}]\right) & \mu_i = \frac{1}{2} \\ \frac{1}{2} g_i & \frac{1}{2} < \mu_i < 1 \\ \frac{1}{2} g_i \frac{Li_{1/2}(e^{-1})}{\sqrt{\pi}} & \mu_i = 1 \end{cases} \tag{4.67}$$

We will have

$$\mathcal{R}_{ki} = \mathcal{R}_{ki}^0 + o(\epsilon_i^{-1/2}).$$

4.2.4 Summary on the results

We have approximation for the integrals in B_{ij} and B_i separately. But we need to add them together to get the total energy for given boundary condition $\cos k\theta$. First of all,

$$
\begin{aligned}
\frac{1}{2}\int_{B_0}|\nabla\phi|^2 &= \frac{1}{2}\sum_{i=1}^{N_B}\int_{B_i}|\nabla\phi|^2 + \frac{1}{2}\sum_{B_{ij}\subset B}|\nabla\phi|^2 \\
&= \frac{1}{2}\sum_{i=1}^{N_B} g_i\left(\mathcal{U}_i - \exp\left[-k\sqrt{\frac{2R\delta_i}{L(L-R)}}\right]\cos k\theta_i\right)^2 \\
&+ \frac{1}{2}\sum_{i=1}^{N_B}(\mathcal{R}_{ki} + k\alpha_i) + \frac{1}{2}\sum_{B_{ij}\subset B} k\alpha_{ij} + O(1) \\
&:= \frac{1}{2}\sum_{i=1}^{N_B} g_i\left(\mathcal{U}_i - S_{ki}\Psi_{ki}^c\right)^2 + \frac{1}{2}\mathcal{R}_k\cdot\mathbf{1} + \frac{1}{2}k\pi + O(1).
\end{aligned}
\tag{4.68}
$$

Where $\Psi_k^c = (\Psi_{k1}^c, \Psi_{k2}^c, \cdots, \Psi_{kN_B}^c)^T \in I\!R^{N_B\times 1}$ is a column vector with i^{th} entry

$$\Psi_{ki}^c = \cos k\theta_i. \tag{4.69}$$

$S_k = \mathrm{diag}\{S_{k1}, S_{k2}, \cdots, S_{kN_B}\} \in I\!R^{N_B\times N_B}$ is the decay matrix with

$$S_{ki} = \exp\left[-k\sqrt{\frac{2R\delta_i}{L(L-R)}}\right] \tag{4.70}$$

It is like an average of the boundary condition near each inclusion D_i and it decays to 0 as k grows.

$$\mathcal{R}_k := (\mathcal{R}_{k1}, \mathcal{R}_{k2}, \cdots, \mathcal{R}_{kN_B}) \in I\!R^{1\times N_B}$$

is the resonance vector with $\mathcal{R}_{ki}$ given by

$$\mathcal{R}_{ki} = \frac{g_i}{2}\left\{\sqrt{\frac{2k\delta_i}{L\pi}}Li_{1/2}\left(\exp\left[-\frac{2k\delta_i}{L}\right]\right) - \exp\left[-2k\sqrt{\frac{2R\delta_i}{(L-R)L}}\right]\right\}. \tag{4.71}$$

Notice that we replaced r_i by $L-R$ and generate a $O(1)$ error which is ignored in this definition.

$$g_i = \pi\sqrt{\frac{2LR}{(L-R)\delta_i}}$$

is the approximation for the effective conductance of the boundary neck B_i.
$\mathbf{1} \in I\!R^{N_B\times 1}$ is a column vector with all entries 1 and

$$\mathcal{R}_k \cdot \mathbf{1} = \sum_{i=1}^{N_B} \mathcal{R}_{ki}.$$

From the partition of B_{ij} and B_i, we will have

$$\sum_{i=1}^{N_B} \alpha_i + \sum_{B_{ij}\subset B} \alpha_{ij} = \pi.$$

Notice that the definition for Ψ_k^c, S_k and $\mathcal{R}_k$ works for all positive integer k. The entries of the decay matrix S_k decays to 0 as k grows.

In the summary, also from the formula (2.49), we have

$$\begin{aligned}\mathcal{E}_p(\cos k\theta) &= \frac{1}{2}\min_{\mathcal{U}}\left\{U^T\Lambda_0^D U + \sum_{\Pi_{ij}\subset B} g_{ij}(\mathcal{U}_i-\mathcal{U}_j)^2 + \sum_{i=1}^{N_B} g_i\left(\mathcal{U}_i - S_{ki}\Psi_{ki}^c\right)^2\right\}[1+O(\sqrt{\frac{\delta}{R}})]\\ &\quad+\frac{1}{2}\left(\mathcal{R}_k\cdot\mathbf{1}+k\pi\right)\\ &=\frac{1}{2}k\pi+\frac{1}{2}\left((S_k\Psi_k^c)^T\Lambda^D(S_k\Psi_k^c)+\mathcal{R}_k\cdot\mathbf{1}\right)[1+O(\sqrt{\frac{\delta}{R}})]\end{aligned} \tag{4.72}$$

Where Λ_0^D, Λ^D are introduced in Section 2.4.3 for our problem.

We summarize the results in this section as the following theorem

Theorem 4.2.4. *For any positive integer k, we have*

$$\langle\cos k\theta\,,\,\Lambda\cos k\theta\rangle = k\pi + \left((S_k\Psi_k^c)^T\Lambda^D(S_k\Psi_k^c)+\mathcal{R}_k\cdot\mathbf{1}\right)[1+O(\sqrt{\frac{\delta}{R}})] \tag{4.73}$$

where Ψ_k^c is defined in (4.69)*,* S_k *is defined in* (4.70)*, and* $\mathcal{R}_k$ *is the resonance term defined in* (4.71)*.* $\Lambda^D \in I\!R^{N_B \times N_B}$ *is the Dirichlet to Neumann map for the discrete resistor network introduced in Section 2.4.3.*

4.3 Approximation with general boundary condition

Our goal is to approximate $\langle \psi(\theta)\, , \Lambda\psi(\theta) \rangle$ for any given boundary data $\psi(\theta)$, where Λ is the Dirichlet to Neumann (DtN) map introduced in (2.5).

For a general boundary condition $\psi(\theta)$, we can always suppose that the media is grounded

$$\int_0^{2\pi} \psi(\theta) = 0,$$

because for any constant ψ, we have $\Lambda\psi = 0$. We would like to write the Fourier expansion for $\psi(\theta)$

$$\psi(\theta) = \sum_{k=1}^{\infty} (a_k^c \cos k\theta + a_k^s \sin k\theta). \tag{4.74}$$

with

$$a_k^c = \frac{1}{\pi} \int_0^{2\pi} \psi(\theta) \cos k\theta d\theta$$
$$a_k^s = \frac{1}{\pi} \int_0^{2\pi} \psi(\theta) \sin k\theta d\theta$$

Because the DtN map Λ is a self adjoint map, it is enough to have the approximation for the following duality pairings

$$
\begin{aligned}
&\langle \cos k\theta \,,\, \Lambda \cos k\theta\rangle, && \text{for all positive integer } k, \\
&\langle \sin k\theta \,,\, \Lambda \sin k\theta\rangle, && \text{for all positive integer } k, \\
&\langle \sin k\theta \,,\, \Lambda \cos m\theta\rangle, && \text{for all positive integer } k, m \\
&\langle \cos k\theta \,,\, \Lambda \cos m\theta\rangle, && \text{for all positive integer } k, m \text{ with } k \neq m \\
&\langle \sin k\theta \,,\, \Lambda \sin m\theta\rangle, && \text{for all positive integer } k, m \text{ with } k \neq m
\end{aligned}
\tag{4.75}
$$

to get the approximation for $\langle \psi(\theta) \,,\, \Lambda\psi(\theta)\rangle$ with any given boundary data $\psi(\theta)$.

However, it is not really necessary to compute all the couples above since we are going to approximate $\langle \psi(\theta) \,,\, \Lambda\psi(\theta)\rangle$. Soppose $\psi(\theta)$ has the expansion (4.74), considering the term with $\sin k\theta$ and $\cos m\theta$, there will be following three terms in the expansion of $\langle \psi(\theta) \,,\, \Lambda\psi(\theta)\rangle$

$$
2a_k^s a_m^c \langle \sin k\theta \,,\, \Lambda \cos m\theta\rangle, \quad (a_k^s)^2 \langle \sin k\theta \,,\, \Lambda \sin k\theta\rangle \text{ and } (a_m^c)^2 \langle \cos m\theta \,,\, \Lambda \cos m\theta\rangle
$$

Since

$$
|2a_k^s a_m^c| \leq (a_k^s)^2 + (a_m^c)^2,
$$

if we can prove

$$
\begin{aligned}
\langle \sin k\theta \,,\, \Lambda \cos m\theta\rangle &\ll \langle \sin k\theta \,,\, \Lambda \sin k\theta\rangle \\
\langle \sin k\theta \,,\, \Lambda \cos m\theta\rangle &\ll \langle \cos m\theta \,,\, \Lambda \cos m\theta\rangle
\end{aligned}
\tag{4.76}
$$

It is all right to make the following approximation

$$\langle \sin k\theta \, , \, \Lambda \cos m\theta \rangle = 0 \tag{4.77}$$

The similar discussion also works for

$$\langle \cos k\theta \, , \, \Lambda \cos m\theta \rangle \text{ and } \langle \sin k\theta \, , \, \Lambda \sin m\theta \rangle, \qquad \text{for all } k \neq m.$$

We already have the approximation for the first duality pairing in (4.75), we will discuss the approximation for the other couples in the following sections.

4.3.1 The approximation for $\langle \sin k\theta \, , \, \Lambda \sin k\theta \rangle$

The case for boundary condition $\sin k\theta$ will be very similar to the case for boundary condition $\cos k\theta$. The discussion for parts inside the domain is the same, and we only need to discuss how to deal with the problem in the boundary layer B. The trial functions for upper and lower bounds are the same in Π_B as we discussed for boundary condition $\cos k\theta$.

In the domain B_0, the trial function for upper bound will be

$$\phi(r, \theta) = w_k(r, \theta) \sin k\theta + w(r, \theta) L(\mathcal{U}) \tag{4.78}$$

which will match the boundary condition $\sin k\theta$ on ∂D. It is also similar to construct $\mathbf{j}$ in B_0 as we discussed for $\cos k\theta$.

In the domains Δ_B, for the upper bound we can still extend the trial functions from Π_B and B_0, such that the whole trial function in B belongs to $H^1(B)$. We also let $\mathbf{j} = 0$ in Δ_B for the lower bound.

Then we will have the same bound for the gap in Π_B and Δ_B as showed in (4.31). For the gap in B_0, it is still 0 in each B_{ij}. In each B_i, we will have

$$\begin{aligned}\Delta\phi &= \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \\ &= (\frac{\partial^2 w_k}{\partial r^2} + \frac{1}{r}\frac{\partial w_k}{\partial r} - \frac{k^2}{r^2}w_k)\sin k\theta + (\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r})\mathcal{U}_i \\ &+ \frac{1}{r^2}\left\{2k\frac{\partial w_k}{\partial\theta}\cos k\theta + \frac{\partial^2 w_k}{\partial\theta^2}\sin k\theta + \frac{\partial^2 w}{\partial\theta^2}\mathcal{U}_i\right\} \\ &= \frac{1}{r^2}\left\{2k\frac{\partial w_k}{\partial\theta}\cos k\theta + \frac{\partial^2 w_k}{\partial\theta^2}\sin k\theta + \frac{\partial^2 w}{\partial\theta^2}\mathcal{U}_i\right\}\end{aligned} \tag{4.79}$$

Hence the gap in B_i is

$$G_{B_i} = \frac{1}{2}\int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} G_i(\theta)d\theta$$

with

$$\begin{aligned}G_i(\theta) &= \int_{L-d(\theta)}^{L} rdr\left\{\frac{1}{r}\int_r^L s\Delta\phi(s,\theta)ds\right\}^2 \\ &= \int_{L-d(\theta)}^{L}\frac{dr}{r}\left\{\int_r^L\frac{ds}{s}\left(2k\frac{\partial w_k(s,\theta)}{\partial\theta}\cos k\theta + \frac{\partial^2 w_k(s,\theta)}{\partial\theta^2}\sin k\theta + \frac{\partial^2 w(s,\theta)}{\partial\theta^2}\mathcal{U}_i\right)\right\}^2 \\ &\le 3G_{i1}^k(\theta) + 3G_{i2}^k(\theta) + 3\mathcal{U}_i^2 G_{i3}(\theta)\end{aligned}$$

where $G_{i1}^k(\theta), G_{i2}^k(\theta), G_{i3}(\theta)$ have the same definition in (4.41) Then we will have the same approximation as in (4.43). And we will have

$$\mathcal{E}_B(\sin k\theta, \mathcal{U}) = \left(\frac{1}{2}\sum_{\Pi_{ij}\subset B} g_{ij}(\mathcal{U}_i - \mathcal{U}_j)^2 + \frac{1}{2}\int_{B_0}|\nabla\phi|^2\right)[1 + O(\sqrt{\frac{\delta}{R}})] \tag{4.80}$$

where ϕ in B_0 is defined in (4.78).

We will have a similar approximation like the discussion for situation with boundary condition $\cos k\theta$,

$$\begin{aligned}\frac{1}{2}\int_{B_0}|\nabla\phi|^2 &= \frac{1}{2}\sum_{i=1}^{N_B}\int_{B_i}|\nabla\phi|^2 + \frac{1}{2}\sum_{B_{ij}\subset B}|\nabla\phi|^2 \\ &= \frac{1}{2}g_i\left(\mathcal{U}_i - S_{ki}\Psi^s_{ki}\right)^2 + \frac{1}{2}\mathcal{R}_k\cdot\mathbf{1} + \frac{1}{2}k\pi + O(1).\end{aligned} \tag{4.81}$$

Where $\Psi^s_k = (\Psi^s_{k1}, \Psi^s_{k2}, \cdots, \Psi^s_{kN_B})^T \in I\!R^{N_B\times 1}$ is a column vector with i^{th} entry

$$\Psi^s_{ki} = \sin k\theta_i. \tag{4.82}$$

And

$$g_i := \pi\sqrt{\frac{2LR}{(L-R)\delta_i}}$$

is an approximation for the effective conductance of the boundary neck B_i defined as before. $\mathcal{R}_k$ is the same resonance vector as before defined in (4.71).

We can use the same way to put $\mathcal{E}_\Pi(\mathcal{U})$ and $\mathcal{E}(\sin k\theta, \mathcal{U})$ together and eliminate $\mathcal{U}$ as the discussion before. In the summary, we will get the following results:

Theorem 4.3.1. *For any positive integer k, we have*

$$\langle \sin k\theta\, ,\, \Lambda \sin k\theta\rangle = k\pi + \left((S_k\Psi^s_k)^T\Lambda^D(S_k\Psi^s_k) + \mathcal{R}_k\cdot\mathbf{1}\right)[1 + O(\sqrt{\frac{\delta}{R}})] \tag{4.83}$$

where Ψ^s_k is defined in (4.82), S_k is defined in (4.70), and $\mathcal{R}_k$ is the resonance term defined in (4.71). $\Lambda^D \in I\!R^{N_B\times N_B}$ is the Dirichlet to Neumann map for the discrete resistor network introduced in Section 2.4.3.

4.3.2 The approximation for $\langle \sin k\theta \, , \, \Lambda \cos m\theta \rangle$

Because Λ is a self adjoint operator, we will have

$$\begin{aligned} 2\langle \sin k\theta \, , \, \Lambda \cos m\theta \rangle &= \langle (\sin k\theta + \cos m\theta) \, , \, \Lambda(\sin k\theta + \cos m\theta) \rangle \\ &\quad - \langle \sin k\theta \, , \, \Lambda \sin k\theta \rangle - \langle \cos m\theta \, , \, \Lambda \cos m\theta \rangle \end{aligned} \tag{4.84}$$

It turns out that we only need to approximate

$$\langle (\sin k\theta + \cos m\theta) \, , \, \Lambda(\sin k\theta + \cos m\theta) \rangle .$$

Since we only change the boundary condition, the key issue is still construct special trial functions in B_0, the discussion for other parts will be exactly the same. We construct it as following

$$\phi(r, \theta) = w_k(r, \theta) \sin k\theta + w_m(r, \theta) \cos m\theta + w(r, \theta) L(\mathcal{U}) \tag{4.85}$$

It will be similar to prove that this trial function in B_0 combining with trial functions in other parts will give us a tight upper bound.

By computing $\nabla\phi$ in B_0, we will have the following expression from what we already have

$$\begin{aligned} &\langle (\sin k\theta + \cos m\theta) \, , \, \Lambda(\sin k\theta + \cos m\theta) \rangle \\ &= (S_k \Psi_k^s + S_m \Psi_m^c)^T \Lambda^D (S_k \Psi_k^s + S_m \Psi_m^c) + (k\pi + m\pi) + (\mathcal{R}_k + \mathcal{R}_m) + O(1) \\ &+ 2 \int_{B_0} \left(\frac{\partial w_k}{\partial r} \frac{\partial w_m}{\partial r} \sin k\theta \cos m\theta - \frac{km}{r^2} w_k w_m \cos k\theta \sin m\theta \right) \\ &- 2 \sum_{i=1}^{N_B} g_i \exp \left[-(k+m) \sqrt{\frac{2R\delta_i}{L(L-R)}} \right] \sin k\theta_i \cos m\theta_i \end{aligned} \tag{4.86}$$

Remember that

$$B_0 = \left(\bigcup B_{ij}\right) \bigcup \left(\bigcup B_i\right).$$

We need to discuss the integration separately in these subdomains like before.

In each B_{ij}, we will have

$$\begin{aligned}
&\int_{B_{ij}} \left(\frac{\partial w_k}{\partial r}\frac{\partial w_m}{\partial r} \sin k\theta \cos m\theta - \frac{km}{r^2} w_k w_m \cos k\theta \sin m\theta \right) \\
&= \int_{\theta_i+\alpha_i}^{\theta_j-\alpha_j} d\theta \int_{L-d}^{L} r dr \left(\frac{\partial w_k}{\partial r}\frac{\partial w_m}{\partial r} \sin k\theta \cos m\theta - \frac{km}{r^2} w_k w_m \cos k\theta \sin m\theta \right) \\
&= \frac{km}{k+m} \int_{\theta_i+\alpha_i}^{\theta_j-\alpha_j} \sin[(k-m)\theta] d\theta + O(1)
\end{aligned} \tag{4.87}$$

The integration in each B_i will be more complicated because it will be singular like the discussion before. When $k = m$, in each B_i, we will have

$$\begin{aligned}
&\int_{B_i} \left(\frac{\partial w_k}{\partial r}\frac{\partial w_m}{\partial r} \sin k\theta \cos m\theta - \frac{km}{r^2} w_k w_m \cos k\theta \sin m\theta \right) \\
&= \int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \sin k\theta \cos k\theta d\theta \int_{L-d}^{L} r dr \left((\frac{\partial w_k}{\partial r})^2 - \frac{k^2}{r^2} w_k^2 \right) \\
&= \int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{-2k^2(1-d/L)^{2k} \ln[1-d/L]}{(1-(1-d/L)^{2k})^2} \sin 2k\theta d\theta \\
&= \frac{g_i}{2} \exp\left[-2k\sqrt{\frac{2R\delta_i}{L(L-R)}} \right] \sin 2k\theta_i + O(1) \\
&= g_i \exp\left[-2k\sqrt{\frac{2R\delta_i}{L(L-R)}} \right] \sin k\theta_i \cos k\theta_i + O(1)..
\end{aligned} \tag{4.88}$$

This just cancel out the last term in (4.86). We will have

$$\begin{aligned}
&\langle (\sin k\theta + \cos m\theta)\,,\, \Lambda(\sin k\theta + \cos m\theta) \rangle = (k\pi + m\pi) \\
&+ \left((S_k \Psi_k^s + S_m \Psi_m^c)^T \Lambda^D (S_k \Psi_k^s + S_m \Psi_m^c) + (\mathcal{R}_k + \mathcal{R}_m) \right) [1 + O(\sqrt{\frac{\delta}{R}})]
\end{aligned}$$

From (4.84) and the results before, we will get

$$\langle \sin k\theta\,,\, \Lambda \cos k\theta \rangle = \left((S_k\Psi_k^s)\Lambda^D(S_k\Psi_k^c)\right)[1 + O(\sqrt{\frac{\delta}{R}})]. \tag{4.89}$$

When $k \neq m$, we will have

$$\begin{aligned}
&\int_{B_i} \left(\frac{\partial w_k}{\partial r}\frac{\partial w_m}{\partial r} \sin k\theta \cos m\theta - \frac{km}{r^2} w_k w_m \cos k\theta \sin m\theta \right) \\
&= \int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} d\theta \int_{L-d}^{L} r dr \left(\frac{\partial w_k}{\partial r}\frac{\partial w_m}{\partial r} \sin k\theta \cos m\theta - \frac{km}{r^2} w_k w_m \cos k\theta \sin m\theta \right) \\
&= \int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{km}{k+m} \sin[(k-m)\theta] d\theta \qquad (4.90) \\
&+ \int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{km}{k+m} \left(\frac{(1-d/L)^{2k}}{1-(1-d/L)^{2k}} + \frac{(1-d/L)^{2m}}{1-(1-d/L)^{2m}} \right) \sin[(k-m)\theta] d\theta \\
&+ \int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{km}{m-k} \left(\frac{1}{1-(1-d/L)^{2k}} - \frac{1}{1-(1-d/L)^{2m}} \right) \sin[(k+m)\theta] d\theta
\end{aligned}$$

In order to have the approximation for $\langle \sin k\theta\,,\, \Lambda \cos m\theta \rangle$ now, we will have the following three different cases.

Case 1

The entries near the main diagonal will be more important than the entries far from the diagonal. When we need to use the matrix as a preconditioner, we are more interested in the block on the main diagonal.

If we use $\cos k\theta, \sin k\theta$ as the basises to approximate this matrix and we want to approximate the entries near the main diagonal, we only need to approximate the duality pairings in (4.75) when $|k-m|$ is small. We suppose that

$$\frac{2|k-m|R}{L} < 1. \tag{4.91}$$

in the following discussion.

When $k \neq m$ and $2|k-m|R < L$, in each B_i, we will have

$$\begin{aligned}
&\int_{B_i} \left(\frac{\partial w_k}{\partial r}\frac{\partial w_m}{\partial r} \sin k\theta \cos m\theta - \frac{km}{r^2} w_k w_m \cos k\theta \sin m\theta \right) \\
&= \frac{km}{k+m} \int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \sin[(k-m)\theta] d\theta \\
&+ \frac{1}{2} g_i \left(\sqrt{\frac{2k\delta_i}{L\pi}} Li_{1/2}(\exp\left[-\frac{2k\delta_i}{L}\right]) \right) \sin[(k-m)\theta_i] \\
&+ \frac{1}{2} g_i \exp\left[-(m+k)\sqrt{\frac{2R\delta_i}{L(L-R)}} \right] \sin[(k+m)\theta_i] + O(1).
\end{aligned} \tag{4.92}$$

Notice that

$$\sum_{B_{ij}} \int_{\theta_i+\alpha_i}^{\theta_j-\alpha_j} \sin[(k-m)\theta] d\theta + \sum_{B_i} \int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \sin[(k-m)\theta] d\theta = \int_0^{2\pi} \sin[(k-m)\theta] d\theta = 0.$$

and

$$\begin{aligned}
&\frac{1}{2} g_i \exp\left[-(m+k)\sqrt{\frac{2R\delta_i}{L(L-R)}} \right] \sin[(k+m)\theta_i] \\
&- g_i \exp\left[-(k+m)\sqrt{\frac{2R\delta_i}{L(L-R)}} \right] \sin k\theta_i \cos m\theta_i \\
&= -\frac{1}{2} g_i \exp\left[-(m+k)\sqrt{\frac{2R\delta_i}{L(L-R)}} \right] \sin[(k-m)\theta_i]
\end{aligned}$$

under the assumption (4.91).

In the summary, we will have the following lemma,

Lemma 4.3.2. *For any positive integer k, m and $2|k-m|R \leq L$,*

$$\langle \sin k\theta \, , \, \Lambda \cos m\theta \rangle = \left((S_k \Psi_k^s)^T \Lambda^D (S_m \Psi_m^c) + \mathcal{R}_k \cdot \Psi_{k-m}^s \right) [1 + O(\sqrt{\frac{\delta}{R}})] \tag{4.93}$$

where Ψ_k^s is defined in (4.82), Ψ_m^c is defined in (4.69) by changing k to m there, and Ψ_{k-m}^s is defined in (4.82) by changing k to $k-m$ there. Similarly, S_k, S_m is defined *in* (4.70)*, and* $\mathcal{R}_k, \mathcal{R}_m$ *is the resonance term defined in* (4.71)*.* $\Lambda^D \in I\!R^{N_B \times N_B}$ *is the Dirichlet to Neumann map for the discrete resistor network introduced in Section 2.4.3.*

In this theorem, there will be no difference for using $\mathcal{R}_k$ or $\mathcal{R}_m$ because k and m are very close. Notice that, we have $\Psi^s_{k-m} = 0$ when $k = m$. The result (4.93) reduced to

$$\langle \sin k\theta \, , \, \Lambda \cos m\theta \rangle = \left((S_k \Psi^s_k)^T \Lambda^D (S_m \Psi^c_m) \right) [1 + O(\sqrt{\frac{\delta}{R}})]$$

which is (4.89).

We already have approximation for entries near the major diagonal. However it is not the whole story, there are some other entries in the matrix. We are going to get the best approximation for these entries as we can.

Case 2

Another case is when

$$\frac{2(k \wedge m)R}{L} \leq O(\sqrt{\frac{R}{\delta}}) \text{ and } 2|k - m|R \geq L.$$

where

$$k \wedge m := \min\{k, m\}.$$

In this case we have

$$\begin{aligned}
&\int_{B_i}\left(\frac{\partial w_k}{\partial r}\frac{\partial w_m}{\partial r}\sin k\theta\cos m\theta-\frac{km}{r^2}w_k w_m\cos k\theta\sin m\theta\right)\\
&=\frac{km}{k+m}\int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i}\sin[(k-m)\theta]d\theta\\
&+\frac{1}{2}g_i\exp\left[-|k-m|\sqrt{\frac{2R\delta_i}{L(L-R)}}\right]\sin[(k-m)\theta_i]\\
&+\frac{1}{2}g_i\exp\left[-(m+k)\sqrt{\frac{2R\delta_i}{L(L-R)}}\right]\sin[(k+m)\theta_i]+O(1).
\end{aligned}\tag{4.94}$$

Like the definition of $\mathcal{R}_k$, We can also define the vector $\mathcal{H}_{km}\in I\!R^{1\times N_B}$ with

$$\begin{aligned}
(\mathcal{H}_{km})_i&=\frac{1}{2}g_i\left(\exp\left[-|k-m|\sqrt{\frac{2R\delta_i}{L(L-R)}}\right]-\exp\left[-(k+m)\sqrt{\frac{2R\delta_i}{L(L-R)}}\right]\right)\\
&=\frac{1}{2}g_i\exp\left[-|k-m|\sqrt{\frac{2R\delta_i}{L(L-R)}}\right]\left(1-\exp\left[-2(k\wedge m)\sqrt{\frac{2R\delta_i}{L(L-R)}}\right]\right)
\end{aligned}\tag{4.95}$$

We will have the following lemma,

Lemma 4.3.3. *For any positive integer k,m, which satisfies $2(k\wedge m)R\leq O(L\sqrt{\frac{R}{\delta}})$ and $2|k-m|R\geq L$,*

$$\langle\sin k\theta\,,\,\Lambda\cos m\theta\rangle=\left((S_k\Psi_k^s)^T\Lambda^D(S_m\Psi_m^c)+\mathcal{H}_{km}\cdot\Psi_{k-m}^s\right)[1+O(\sqrt{\frac{\delta}{R}})]\tag{4.96}$$

where $\mathcal{H}_{km}$ is defined in (4.95).

Case 3

The last case is

$$\frac{2(k\wedge m)R}{L}\gg O(\sqrt{\frac{R}{\delta}})\text{ and }2|k-m|R\geq L.$$

In this case, we have

$$\langle \sin k\theta\,,\,\Lambda \sin k\theta\rangle = k\pi[1+o(1)] \gg O(\sqrt{\frac{R}{\delta}})$$
$$\langle \cos m\theta\,,\,\Lambda \cos m\theta\rangle = m\pi[1+o(1)] \gg O(\sqrt{\frac{R}{\delta}})$$

It is easy to show that

$$\langle \sin k\theta\,,\,\Lambda \cos m\theta\rangle \leq O(\sqrt{\frac{R}{\delta}}) \tag{4.97}$$

which satisfies the condition in (4.76). It means we can let

$$\langle \sin k\theta\,,\,\Lambda \cos m\theta\rangle = 0 \tag{4.98}$$

in this case for our approximation.

At the end, we summarize the results in this section as a theorem

Theorem 4.3.4. *For any positive integer k, m and $2|k-m|R \leq L$,*

$$\langle \sin k\theta\,,\,\Lambda \cos m\theta\rangle = \left((S_k\Psi_k^s)^T\Lambda^D(S_m\Psi_m^c) + \mathcal{R}_k\cdot\Psi_{k-m}^s\right)[1+O(\sqrt{\frac{\delta}{R}})]$$

For positive integer k, m, which satisfies $2(k\wedge m)R \leq O(L\sqrt{\frac{R}{\delta}})$ and $2|k-m|R \geq L$,

$$\langle \sin k\theta\,,\,\Lambda \cos m\theta\rangle = \left((S_k\Psi_k^s)^T\Lambda^D(S_m\Psi_m^c) + \mathcal{H}_{km}\cdot\Psi_{k-m}^s\right)[1+O(\sqrt{\frac{\delta}{R}})]$$

For positive integer k, m, which satisfies $2(k\wedge m)R \gg O(L\sqrt{\frac{R}{\delta}})$ and $2|k-m|R \geq L$,

$$\langle \sin k\theta\,,\,\Lambda \cos m\theta\rangle = \min\{\langle \sin k\theta\,,\,\Lambda \sin k\theta\rangle, \langle \cos m\theta\,,\,\Lambda \cos m\theta\rangle\}\cdot o(1)$$

where Ψ_k^s, Ψ_{k-m}^s *is defined in* (4.82) *and* Ψ_m^c *is defined in* (4.69)*.* S_k, S_m *is defined in* (4.70)*.* $\mathcal{R}_k, \mathcal{R}_m$ *is the resonance term defined in* (4.71) *and* $\mathcal{H}_{km}$ *is defined in* (4.95)*.* $\Lambda^D \in I\!R^{N_B \times N_B}$ *is the Dirichlet to Neumann map for the discrete resistor network introduced in Section 2.4.3.*

Remark 4.3.5. *In the first two cases of the above theorem, we basically have all the singular terms with an* $O(1)$ *error. This is the best we can do, since our variational method will generate an* $O(1)$ *error, which is the gap between the upper and lower bounds. In the third case, it is a hard region for us to have an approximation with an* $O(1)$ *error. What we did is to keep the leading order but ignore other terms.*

4.3.3 The approximation for $\langle \cos k\theta\,,\, \Lambda \cos m\theta \rangle$ and $\langle \sin k\theta\,,\, \Lambda \sin m\theta \rangle$

Similarly, we will have results for $\langle \cos k\theta\,,\, \Lambda \cos m\theta \rangle$ and $\langle \sin k\theta\,,\, \Lambda \sin m\theta \rangle$. The discussion will be almost the same as the discussion for $\langle \sin k\theta\,,\, \Lambda \cos m\theta \rangle$. In this section, we will directly state the theorems for these two couples.

Theorem 4.3.6. *For positive integer* $k \neq m$ *and* $2|k-m|R \leq L$*,*

$$\langle \cos k\theta\,,\, \Lambda \cos m\theta \rangle = \left((S_k \Psi_k^c)^T \Lambda^D (S_m \Psi_m^c) + \mathcal{R}_k \cdot \Psi_{k-m}^c\right)[1 + O(\sqrt{\frac{\delta}{R}})]$$

For positive integer k, m*, which satisfies* $2(k \wedge m)R \leq O(L\sqrt{\frac{R}{\delta}})$ *and* $2|k-m|R \geq L$*,*

$$\langle \cos k\theta\,,\, \Lambda \cos m\theta \rangle = \left((S_k \Psi_k^c)^T \Lambda^D (S_m \Psi_m^c) + \mathcal{H}_{km} \cdot \Psi_{k-m}^c\right)[1 + O(\sqrt{\frac{\delta}{R}})]$$

For positive integer k, m*, which satisfies* $2(k \wedge m)R \gg O(L\sqrt{\frac{R}{\delta}})$ *and* $2|k-m|R \geq L$*,*

$$\langle \cos k\theta\,,\, \Lambda \cos m\theta \rangle = \min\{\langle \cos k\theta\,,\, \Lambda \cos k\theta \rangle, \langle \cos m\theta\,,\, \Lambda \cos m\theta \rangle\} \cdot o(1)$$

where $\Psi^c_k, \Psi^c_m, \Psi^c_{k-m}$*, is defined in* (4.69)*.* S_k, S_m *is defined in* (4.70)*.* $\mathcal{R}_k, \mathcal{R}_m$ *is the resonance term defined in* (4.71) *and* $\mathcal{H}_{km}$ *is defined in* (4.95)*.* $\Lambda^D \in I\!R^{N_B \times N_B}$ *is the Dirichlet to Neumann map for the discrete resistor network introduced in Section 2.4.3.*

Theorem 4.3.7. *For positive integer* $k \neq m$ *and* $2|k-m|R \leq L$*,*

$$\langle \sin k\theta \,,\, \Lambda \sin m\theta \rangle = \big((S_k \Psi^s_k)^T \Lambda^D (S_m \Psi^s_m) + \mathcal{R}_k \cdot \Psi^c_{k-m}\big)\, [1 + O(\sqrt{\frac{\delta}{R}})]$$

For positive integer k, m*, which satisfies* $2(k \wedge m)R \leq O(L\sqrt{\frac{R}{\delta}})$ *and* $2|k-m|R \geq L$*,*

$$\langle \sin k\theta \,,\, \Lambda \sin m\theta \rangle = \big((S_k \Psi^s_k)^T \Lambda^D (S_m \Psi^s_m) + \mathcal{H}_{km} \cdot \Psi^c_{k-m}\big)\, [1 + O(\sqrt{\frac{\delta}{R}})]$$

For positive integer k, m*, which satisfies* $2(k \wedge m)R \gg O(L\sqrt{\frac{R}{\delta}})$ *and* $2|k-m|R \geq L$*,*

$$\langle \sin k\theta \,,\, \Lambda \sin m\theta \rangle = \min\{\langle \sin k\theta \,,\, \Lambda \sin k\theta \rangle, \langle \sin m\theta \,,\, \Lambda \sin m\theta \rangle\} \cdot o(1)$$

where Ψ^s_k, Ψ^s_m *is defined in* (4.82) *and* Ψ^c_{k-m} *is defined in* (4.69)*.* S_k, S_m *is defined in* (4.70)*.* $\mathcal{R}_k, \mathcal{R}_m$ *is the resonance term defined in* (4.71) *and* $\mathcal{H}_{km}$ *is defined in* (4.95)*.* $\Lambda^D \in I\!R^{N_B \times N_B}$ *is the Dirichlet to Neumann map for the discrete resistor network introduced in Section 2.4.3.*

Chapter 5

Summary and future work

5.1 Summary

In this thesis, we use variational methods to approximate the energy for a high contrast elliptic problem with any boundary condition. Since the oscillation of the boundary condition will have some effect in the total energy and the effect will not go far from the boundary, we divide our problem into two problems, which are located in two separated subdomains. We use existing results to approximate the energy in the subdomain far from the boundary, and develop a way to approximate the energy in the area near the boundary. Then we combine these two results and get our approximation for energy with arbitrary boundary condition. In other words, we approximated the Dirichlet to Neumann (DtN) map for the problem in high contrast media.

More precisely, we can use an approximation matrix up to any size to approximate the continuous DtN map for the high contrast two phase composites. In our approximation, we basically captured the leading order $O(\sqrt{\frac{R}{\delta}})$ of the DtN map, It is a singular term because that the distance δ between neighbor inclusions and the radius R of inclusions are in different scales.

In numerical methods, if we want simulatie the flux for this problem, we need the mesh size $h < \delta$. We will end up with a very huge linear system and it may be very difficult to solve. The idea is to use our approximation of the DtN map as a preconditioner in our numerical method, such that we can solve problems numerically in high contrast media more efficient.

5.2 Application to domain decomposition methods

In this section, we will describe how to apply the results we have obtained to develop fast domain decomposition methods. Because we have the approximation for the Dirichlet to Neumann map, it will be a good idea to use it as a preconditioner in our numerical methods.

5.2.1 The problem

We are considering the following elliptic problem in the domain $\Omega \in I\!R^2$:

$$\begin{aligned} -\nabla \cdot \sigma(\mathbf{x})\nabla u(\mathbf{x}) &= f, \quad \Omega \\ u(\mathbf{x}) &= 0, \quad \partial\Omega \end{aligned} \tag{5.1}$$

Suppose Ω is partitioned into two nonoverlapping subdomains Ω_1, Ω_2, where $\Omega = \overline{\Omega_1} \cup \Omega_2, \Omega_1 \cap \Omega_2 = \emptyset, \Gamma := \partial\Omega_1 \cap \partial\Omega_2$. For simplicity in our problem, we suppose that $\Omega = B(0,2), \Omega_1 = B(0,1), \Omega_2 = B(0,2) \setminus \overline{B}(0,1), \Gamma = \partial B(0,1)$, see Figure 5.1.

We also suppose that $\sigma(\mathbf{x})$ only has high contrast values in Ω_1, but it doesn't have high contrast values in Ω_2. In other words, $\sigma(\mathbf{x})$ is huge in the inclusions, but it is

$O(1)$ in the other places, see Figure 5.1.

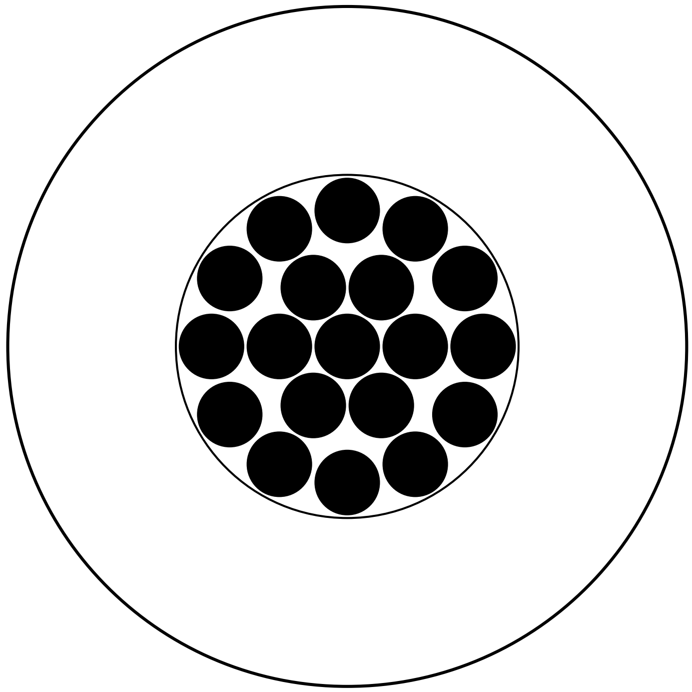


Figure 5.1: The problem in high contrast domain, the black disks are inclusions.

To present our idea, we only discuss the case for two subdomains here. However it is easy to generalize our methods to the problem with many subdomains. In each subdomain which contains inclusions, it should satisfy the geometric assumptions we made in Section 2.4.

Now we have a high contrast problem. We know that the solution for (1.1) will change a lot in the necks between different inclusions, and it is smooth in other places of Ω. In order to catch the big changes of the solution in these necks, we need very fine mesh to solve the problem numerically.

In this sense, we will have a very large linear system

$$Au = f. \tag{5.2}$$

It will be very difficult to solve this problem directly, because the matrix A may have huge size and it may be ill conditioned . So we are interested in the domain decomposition methods for our problem, see [13, 25] and refers therein.

5.2.2 Circular grids and finite volume discretization

In the last section, we didn't describe what kind of numerical methods we will use to get the linear system (5.2). We will use a finite volume discretization for our problem (5.1) on a circular grids, see [9, 10, 11].

In this discretization, there will be M vertexes on the interface Γ. In order to compute the flux in the necks between inclusions, the mesh size must be smaller than δ, which is the distance between neighbor inclusions. It means that $M \geq O(1/\delta) \gg N_B$, where N_B is the number of inclusions near the interface Γ in Ω_1.

Remark 5.2.1. *I will give more details later.*

5.2.3 Nonoverlapping domain decomposition methods

The idea of nonoverlapping domain decomposition methods is to split a problem in a big domain into many subproblems in small subdomains. It will use the information on the interface to communicate with each other subdomain. If we can have some information on the interface, trace of the solution or the flux, we can use it as Dirichlet condition or Neumann condition for the problems in subdomains. Then we only need to solve a small linear system in each subdomain to get the solution over the whole domain. There are many advantages to solve some small systems rather than to solve a huge linear system.

The key issue of nonoverlapping domain decomposition methods is to solve some information on the interface first. It could be the trace of the solution or the flux on the interface. After we have the information on the interface, we can use it as a Dirichlet or Neumann condition to solve problems in each subdomain separately. In general, we will have interface equations to solve the trace of the solution or the flux on the interface.

In this section, we will introduce nonoverlapping domain decomposition methods for two subdomains, and introduce two different kinds of interface equations. We will follow the idea of Toselli and Widlund's book [25].

Base on the partition introduced in Section 5.2.1, we can write the linear system into the following block form:

$$\begin{bmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}. \tag{5.3}$$

where we divide the degrees of freedom into Ω_1,Ω_2 and Γ, respectively. The blocks A_{12}, A_{21} are zero only under the assumption that the nodes in Ω_1 and Ω_2 are not directly coupled. Since we will use finite volume methods for our problem, they are not directly coupled.

There are inclusions and different scales inside the domain Ω_1 and the solution in Ω_1 will change fast in some places, we need really fine mesh inside Ω_1. In this sense, the submatrix A_{11} will have very large size. We need to avoid solving linear systems in Ω_1 as much as we can. But we can use very coarse mesh in Ω_2 because there is no singularity in Ω_2. The matrix A_{22} will be relatively small and it will not be so expensive to solve linear systems in Ω_2. A_{33} is matrix belongs to $\mathbb{R}^{M\times M}$.

The Schur complement system

If u_3 is known, from the equation (5.3) we have

$$
\begin{aligned}
u_1 &= A_{11}^{-1}(f_1 - A_{13}u_3),\\
u_2 &= A_{22}^{-1}(f_2 - A_{23}u_3).
\end{aligned}
\tag{5.4}
$$

Substituting for u_1, u_2 in the equation (5.3), we have a reduced problem for the unknown u_3

$$
Su_3 = g, \tag{5.5}
$$

where

$$
\begin{aligned}
S &= A_{33} - A_{13}^T A_{11}^{-1} A_{13} - A_{23}^T A_{22}^{-1} A_{23}\\
g &= f_3 - A_{13}^T A_{11}^{-1} f_1 - A_{23}^T A_{22}^{-1} f_2.
\end{aligned}
\tag{5.6}
$$

This is equation for solving the trace of the solution on the interface Γ. After solving the problem (5.5), we only need to solve a linear system once in each subdomain to get u, see (5.4).

The matrix $S \in I\!R^{M\times M}$ is the Schur complement of A_{33} in A, where M is the number of discretization nodes on the interface Γ. S is very expansive to compute, it requires M solves in each subdomain because we need to compute $A_{ii}^{-1}A_{i3}$, see (5.6). S is also a dense matrix, it will be difficult to solve (5.5) directly even if we have the matrix S.

In general, we will use interface preconditioners to solve the system (5.5) without computing S explicitly. With a good preconditioner, we have a big chance to get u_3 by using far less than M solves in each subdomains.

The main idea is to write S into the sum of two parts which reflect the contribution from Ω_1 and Ω_2 more explicitly. The term A_{33} can be written as

$$A_{33} = A_{33}^{(1)} + A_{33}^{(2)},$$

where $A_{33}^{(i)}$ corresponds to the contribution to A_{33} from the subdomain Ω_i.

In this case, we can write

$$S = S^{(1)} + S^{(2)},$$

where

$$S^{(i)} = A_{33}^{(i)} - A_{i3}^T A_{ii}^{-1} A_{i3}, \quad i = 1, 2. \tag{5.7}$$

By this way, we can also split f_3

$$f_3 = f_3^{(1)} + f_3^{(2)},$$

and define

$$g^{(i)} = f_3^{(i)} - A_{i3}^T A_{ii}^{-1} f_i. \tag{5.8}$$

In general, the preconditioner for (5.5) will be $S^{(1)-1}$,$S^{(2)-1}$ or the combination of these two. $S^{(i)-1}$ is the discretized Neumann to Dirichlet map for the subproblem in Ω_i on the interface Γ, see [13].

Considering we have the approximation for the Dirichlet to Newmann map in our previous discussion. We will first solve out the flux on the interface, which is a solution of a flux equation. We can use our approximation of the Dirichlet to Neumann map as a precondition for the flux equation.

The equation for flux on interface

Now suppose $\gamma = \gamma^{(1)} = -\gamma^{(2)}$ is the flux on Γ which points from Ω_1 to Ω_2. If we see $\gamma^{(i)}$ as a Neumann condition on the interface Γ and considering

$$\begin{aligned} -\nabla \cdot (\sigma \nabla u_i) &= f_i \qquad \text{in } \Omega_i, \\ u_i &= 0 \qquad \text{on } \partial\Omega_i \setminus \Gamma, \\ \frac{\partial u_i}{\partial n_i} &= \gamma^{(i)} \qquad \text{on } \Gamma, \end{aligned} \tag{5.9}$$

We will have the following equations in each subdomain Ω_i,

$$\begin{bmatrix} A_{ii} & A_{i3} \\ A_{3i} & A_{33}^{(i)} \end{bmatrix} \begin{bmatrix} u_i \\ u_3^{(i)} \end{bmatrix} = \begin{bmatrix} f_i \\ f_3^{(i)} + \gamma^{(i)} \end{bmatrix} \tag{5.10}$$

It will give us

$$u_3^{(i)} = S^{(i)-1}(g^{(i)} + \gamma^{(i)}),$$

Then the equation for γ on Γ is to ensure that

$$u_3^{(1)} = u_3^{(2)} \qquad \text{on } \Gamma.$$

We will have the following equation for flux on Γ

$$F\gamma = d, \qquad \text{on } \Gamma \tag{5.11}$$

where

$$
\begin{aligned}
F &= S^{(1)-1} + S^{(2)-1} \\
d &= d^{(1)} + d^{(2)} = -S^{(1)-1} g^{(1)} + S^{(2)-1} g^{(2)}.
\end{aligned}
$$

with $g^{(i)}$ is defined in (5.8).

In this case, it is natural to use $S^{(1)}, S^{(2)}$ or their combinations as the preconditioner to solve the system (5.11). $S^{(i)}$ is the discretized Dirichlet to Neumann map of the subproblem in Ω_i along the interface Γ. Since we use it as a preconditioner, we can use an approximation for the Dirichlet to Neumann map instead of the exact $S^{(i)}$.

This section gave us the idea of domain decomposition methods in matrix forms, but we don't really want to compute the matrix S or F to solve the systems (5.6) or (5.11). People usually use iterative methods to solve the equations on the interface. We also explained why we prefer to solve the system (5.11) instead of (5.6).

In the next section, we will introduce an iterative method without computing F in (5.11) explicitly. We will also use our approximation of the Dirichlet to Neumann map in Ω_1 as an alternative preconditioner.

5.2.4 The modified Dirichlet-Dirichlet algorithm

Following Toselli and Widlund's book [25], we will first introduce the Dirichlet-Dirichlet algorithm. Then we will introduce our modified algorithm, which will use our approximation for the Dirichlet to Neumann map in the high contrast subdomain Ω_1.

The Dirichlet-Dirichlet algorithm

Assume $\gamma^n = \gamma_1^n = -\gamma_2^n$ is flux on Γ in the n^{th} iteration. In this iteration, we need to update γ^n to γ^{n+1} in some way. The Dirichlet-Dirichlet algorithm will have the following three steps in each iteration.

1. The first step is to solve a Neumann problem in each subdomain:

$$\begin{aligned} -\nabla \cdot (\sigma \nabla u_i^{n+1/2}) &= f_i \qquad \text{in } \Omega_i, \\ u_i^{n+1/2} &= 0 \qquad \text{on } \partial\Omega_i \setminus \Gamma, \\ \frac{\partial u_i^{n+1/2}}{\partial n_i} &= \gamma_i^n \qquad \text{on } \Gamma, \end{aligned} \tag{5.12}$$

2. The second step is to solve a Dirichlet problem in each subdomain

$$\begin{aligned} -\nabla \cdot (\sigma \nabla v_i^{n+1}) &= 0 \qquad \text{in } \Omega_i, \\ v_i^{n+1} &= 0 \qquad \text{on } \partial\Omega_i \setminus \Gamma, \\ v_i^{n+1} &= u_1^{n+1/2} - u_2^{n+1/2} \qquad \text{on } \Gamma, \end{aligned} \tag{5.13}$$

3. The third step is to correct γ^n

$$\gamma^{n+1} = \gamma^n - \theta\left(\frac{\partial v_1^{n+1}}{\partial n_1} + \frac{\partial v_2^{n+1}}{\partial n_2}\right). \tag{5.14}$$

with a suitable $\theta \in (0, \bar{\theta})$.

If we write this iteration into the matrix form, we will have

$$\gamma^{n+1} - \gamma^n = \theta(S^{(1)} + S^{(2)})(d - F\gamma^n), \tag{5.15}$$

which is a preconditioned Richardson iteration for the system (5.11) with the preconditioner $S^{(1)} + S^{(2)}$.

The modified Dirichlet-Dirichlet algorithm

However in our problem, we will use very fine mesh in Ω_1 to ensure the accurace of the numerical solution. This motivate us to avoid solving subproblems in Ω_1, or solve less subproblems in Ω_1. The idea is to use the approximation for the Dirichlet to Neumann map we obtained before. Actually, we can have a matrix up to any size, which is an approximation for the Dirichlet to Neumann map. Here we use a matrix $\Lambda_M \in I\!R^{M\times M}$ as an approximation of the Dirichlet to Neumann map in Ω_1 on Γ.

When we discretize the problem and trying to solve it numerically, we will have a discretized Dirichlet to Neumann map $S^{(1)} \in I\!R^{M\times M}$ in Ω_1. It is natural to use Λ_M to approximate the $S^{(1)}$.

We can now modify the Dirichlet-Dirichlet algorithm and use our approximation there. Assume $\gamma^n = \gamma_1^n = -\gamma_2^n$ is flux on Γ in the n^{th} iteration. The three steps in n^{th} iteration will be

1. The first step is still to solve a Neumann problem in each subdomain:

$$
\begin{aligned}
-\nabla \cdot (\sigma \nabla u_i^{n+1/2}) &= f_i \qquad \text{in } \Omega_i, \\
u_i^{n+1/2} &= 0 \qquad \text{on } \partial\Omega_i \setminus \Gamma, \\
\frac{\partial u_i^{n+1/2}}{\partial n_i} &= \gamma_i^n \qquad \text{on } \Gamma,
\end{aligned}
\tag{5.16}
$$

2. The second step is to solve a Dirichlet problem in Ω_2 only

$$
\begin{aligned}
-\Delta v_2^{n+1} &= 0 \qquad \text{in } \Omega_2,\\
v_2^{n+1} &= 0 \qquad \text{on } \partial\Omega, \\
v_2^{n+1} &= u_1^{n+1/2} - u_2^{n+1/2} \qquad \text{on } \Gamma,
\end{aligned}
\tag{5.17}
$$

3. The third step is to correct γ^n

$$\gamma^{n+1} = \gamma^n - \theta \left[\Lambda_M (u_1^{n+1/2} - u_2^{n+1/2}) + \frac{\partial v_2^{n+1}}{\partial n_2} \right]. \tag{5.18}$$

with a suitable $\theta \in (0, \bar{\theta})$.

In this modified algorithm, we only solve a half number of subproblems in Ω_1 as in the original Dirichlet-Dirichlet algorithm, if they have the same number of iterations to converge to the true solution.

If we write this new iteration into the matrix form, we will have

$$\gamma^{n+1} - \gamma^n = \theta(\Lambda_M + S^{(2)})(d - F\gamma^n), \tag{5.19}$$

which is a preconditioned Richardson iteration for the system (5.11) with the preconditioner $\Lambda_M + S^{(2)}$.

We need to prove the following lemma

Lemma 5.2.2. *The condition number of the matrix* $(\Lambda_M + S^{(2)})(S^{(1)-1} + S^{(2)-1})$ *will not depend on neither the contrast of the media or the size of the mesh.*

This lemma means that our modified algorithm will have similar iteration steps as the Dirichlet-Dirichlet algorithm to converge to the true solutions.

5.2.5 A fast way to get an initial guess

In the iterative methods, a good initial guess is really important. Still remember that we want to avoid solving subproblems in Ω_1. In this section, we will see that in order to get an initial guess, we don't need to solve any subproblem in Ω_1 at all.

Suppose γ_n^0 is the flux on Γ which points from Ω_1 to Ω_2 in the n^{th} iteration to get an initial guess γ^0. Here we can start from $\gamma_0^0 = 0$. There are two steps in each iteration.

1. The first step is to solve a Neumann problem in Ω_2:

$$\begin{aligned} -\Delta u_2^{n+1/2} &= f_2 \qquad \text{in } \Omega_2, \\ u_2^{n+1/2} &= 0 \qquad \text{on } \partial\Omega \\ \frac{\partial u_2^{n+1/2}}{\partial n_2} &= -\gamma_n^0 \qquad \text{on } \Gamma, \end{aligned} \tag{5.20}$$

2. Then we correct γ_n^0 like following

$$\gamma_{n+1}^0 = (1-\theta)\gamma_n^0 + \theta(\Lambda_M u_2^{n+1/2}). \tag{5.21}$$

with a suitable $\theta \in (0, \bar{\theta})$.

If Λ_M in the iteration step (5.21) is the exact discretized Dirichlet to Neumann map $S^{(1)}$. The iteration steps above in matrix form will be

$$\gamma_{n+1}^0 - \gamma_n^0 = \theta S^{(1)}(d - F\gamma_n^0). \tag{5.22}$$

which is a preconditioned Richardson iteration for the system (5.11) with the preconditioner $S^{(1)}$. It will converge to the solution of the system (5.11).

When we use the approximation Λ_M for $S^{(1)}$ like showed in (5.21), the convergence vector γ^0 will be an approximation for the solution of (5.11). It will be a good choice for the initial guess in our modified Dirichlet-Dirichlet method. Notice that in the iteration steps to get γ^0, we didn't solve any subproblem in Ω_1.

Appendix A

Functional analysis

A.1 Euler-Lagrange Equations

Suppose $\Omega \in \mathbb{R}^2$ is a domain with some holes $D_i (i \in \mathcal{S})$ inside. The boundary of Ω is

$$\partial\Omega = \Gamma_D \cup \Gamma_N \cup (\cup_{i \in \mathcal{S}} \partial D_i) \tag{A.1}$$

where $|\Gamma_D| > 0$.

We are considering the following minimization problem

$$\mathcal{E} = \min_{\phi \in V} \frac{1}{2} \int_{\Omega} |\nabla \phi|^2, \tag{A.2}$$

where

$$V = \{\phi \in H^1(\Omega) : \phi|_{\Gamma_D} = \psi, \phi|_{\partial D_i} = \text{constant } , \forall i \in \mathcal{S}\}. \tag{A.3}$$

Define another space

$$V_0 = \{\phi \in H^1(\Omega) : \phi|_{\Gamma_D} = 0, \phi|_{\partial D_i} = \text{constant } , \forall i \in \mathcal{S}\}. \tag{A.4}$$

Then for any $u \in V, v \in V_0$ and $s \in I\!R$, we have $u + sv \in V$.

Suppose u is the minimizer of the problem (A.2), we are going to find what condition should the minimizer u satisfy.

Let

$$F[\phi] = \frac{1}{2}\int_\Omega |\nabla\phi|^2, \tag{A.5}$$

and

$$f(s) = F[u + sv] = \frac{1}{2}\int_\Omega |\nabla(u + sv)|^2 \tag{A.6}$$

with $u \in V, v \in V_0$ and $s \in I\!R$.

u is the minimizer of the problem (A.2) means $f'(0) = 0$. From (A.6),

$$f'(0) = \int_\Omega \nabla u \cdot \nabla v,$$

Then we will have

$$\int_\Omega \nabla u \cdot \nabla v = 0, \qquad \text{for all } v \in V_0 \tag{A.7}$$

From Green's theorem, for all $v \in V_0$,

$$
\begin{aligned}
0 &= -\int_\Omega \Delta u v + \int_{\Gamma_D} \frac{\partial u}{\partial n} v + \int_{\Gamma_N} \frac{\partial u}{\partial n} v + \sum_{i \in \mathcal{S}} \int_{\partial D_i} \frac{\partial u}{\partial n} v \\
&= -\int_\Omega \Delta u v + \int_{\Gamma_N} \frac{\partial u}{\partial n} v + \sum_{i \in \mathcal{S}} v|_{\partial D_i} \int_{\partial D_i} \frac{\partial u}{\partial n}
\end{aligned}
\tag{A.8}
$$

Then we will have the following equations for u

$$
\begin{aligned}
\Delta u &= 0, && \text{in } \Omega \\
\frac{\partial u}{\partial n} &= 0, && \text{on } \Gamma_N \\
\int_{\partial D_i} \frac{\partial u}{\partial n} &= 0, && \text{on } \partial D_i, \forall i \in \mathcal{S}
\end{aligned}
\tag{A.9}
$$

Plus the constraint conditions in the space V, we will have the Euler-Lagrange equations for the minimizer of the problem (A.2)

$$
\begin{aligned}
\Delta u &= 0, && \text{in } \Omega \\
u &= t_i, && \text{on } \partial D_i, \forall i \in \mathcal{S} \\
\frac{\partial u}{\partial n} &= 0, && \text{on } \Gamma_N \\
\int_{\partial D_i} \frac{\partial u}{\partial n} &= 0, \forall i \in \mathcal{S} \\
u &= \psi, && \text{on } \Gamma_D
\end{aligned}
\tag{A.10}
$$

where t_i are constants need to be determined from the above equations. We denote the solution of the problem (A.10) as $(u, \mathcal{T})$, where $\mathcal{T} = (t_1, t_2, \cdots)^T$ is a vector of potentials on $\partial D_i (i \in \mathcal{S})$.

When there is no holes D_i inside the domain, the Euler-Lagrange equations will be

$$\begin{aligned} \Delta u &= 0, && \text{in } \Omega \\ \frac{\partial u}{\partial n} &= 0, && \text{on } \Gamma_N \\ u &= \psi, && \text{on } \Gamma_D \end{aligned} \tag{A.11}$$

A.2 Uniqueness and maximal principle

In this section, we are going to prove two lemmas related to the Euler-Lagrange equations.

Lemma A.2.1 (Uniqueness)**.** *The Euler-Lagrange equations* (A.10) *have an unique solution.*

Proof. Suppose there are two different solutions of the problem (A.10), they are $(u_1, \mathcal{T}_1)$ and $(u_2, \mathcal{T}_2)$. Then $(u, \mathcal{T}) = (u_1 - u_2, \mathcal{T}_1 - \mathcal{T}_2)$ will be solution of the following problem

$$\begin{aligned} \Delta u &= 0, && \text{in } \Omega \\ u &= t_i, && \text{on } \partial D_i \\ \frac{\partial u}{\partial n} &= 0, && \text{on } \Gamma_N \\ \int_{\partial D_i} \frac{\partial u}{\partial n} &= 0, && \text{on } \partial D_i, \forall i \in \mathcal{S} \\ u &= 0, && \text{on } \Gamma_D \end{aligned} \tag{A.12}$$

Then

$$\int_\Omega |\nabla u|^2 = -\int_\Omega (\Delta u) u + \int_{\Gamma_D} \frac{\partial u}{\partial n} u + \int_{\Gamma_N} \frac{\partial u}{\partial n} u + \sum_{i \in \mathcal{S}} \int_{\partial D_i} \frac{\partial u}{\partial n} u = 0. \tag{A.13}$$

This means u is a constant in Ω. From the continuity, we have $u = 0$ and $\mathcal{T} = 0$. So the Euler-Lagrange equations (A.10) have an unique solution $(u, \mathcal{T})$. The uniqueness argument is also true for the equation (A.11). □

Lemma A.2.2 (Maximum principle)**.** *Let* $(u, \mathcal{T})$ *be the solution of the problem* (A.10)*, then*

$$\max_{\bar{\Omega}}\{u\} \leq \max_{\Gamma_D}\{\psi\} \text{ and } \max_i\{t_i\} \leq \max_{\Gamma_D}\{\psi\} \tag{A.14}$$

Proof. Suppose $M := \max_{\bar{\Omega}} u$, and let $A := \{x \in \Omega : u(x) = M\}$. A is relatively closed in Ω.

If there is a $x_0 \in \Omega$ such that $u(x_0) = M$, from mean value theorem, there will be a small enough r such that $B(x_0, r) \subset A$.

If there is a $t_i = M$, suppose the radius of the hole D_i is R_i and it's center is x_0. We will have a similar mean value theorem

$$M = \frac{1}{\pi(R_i + r)^2 - \pi R_i^2} \int_{B(x_0, R_i + r) \setminus \overline{B(x_0, R_i)}} u \leq M$$

for small enough r. Then we will have

$$B(x_0, R_i + r) \setminus \overline{B(x_0, R_i)} \subset A, \text{ for some } r > 0$$

So A is relative open in either situation above. Since Ω is simple connected, we have $A = \Omega$. It means the solution will be a constant if either situation above happened. Until now, we proved the maximum principle for the equation (A.10). □

From this lemma, we see that $\mathcal{U}$ in Lemma 3.3.1 cannot be arbitrary, it is bounded by the boundary condition ψ.

A.3 Legendre Transformation

We are still considering the minimization problem (A.2) with the test space V in (A.3). In this section we are going to get the dual problem of the minimization problem (A.2), which is a maximization problem.

First let us introduce the following space

$$W = \{\mathbf{j} \in \mathbf{L}^2(\Omega) : \nabla \cdot \mathbf{j} = 0, \int_{D_i} \mathbf{j} \cdot \mathbf{n} = 0 (\forall i \in \mathcal{S}), \mathbf{j} \cdot \mathbf{n}|_{\Gamma_N} = 0\}. \tag{A.15}$$

The derivative of the trial functions $\mathbf{j}$ is in week sense, for example $\nabla \cdot \mathbf{j} = 0$ means

$$\int_\Omega \nabla\phi \cdot \mathbf{j} = -\int_\Omega \nabla \cdot \mathbf{j}\phi = 0, \qquad \text{for all } \phi \in C_0^\infty(\Omega).$$

The Legendre transformation for any vector $\mathbf{v} \in I\!R^2$ is

$$\frac{1}{2}\mathbf{v}^2 = \max_{\mathbf{j} \in I\!R^2}(\mathbf{v}\mathbf{j} - \frac{1}{2}\mathbf{j}^2) \tag{A.16}$$

This is true because

$$\max_{\mathbf{j} \in I\!R^2}(\mathbf{v}\mathbf{j} - \frac{1}{2}\mathbf{j}^2) = \frac{1}{2}\mathbf{v}^2 - \frac{1}{2}\max_{\mathbf{j} \in I\!R^2}(\mathbf{v} - \mathbf{j})^2 = \frac{1}{2}\mathbf{v}^2.$$

Then (A.2) becomes

$$\begin{aligned} \mathcal{E} &= \min_{\phi \in V} \frac{1}{2} \int_\Omega |\nabla\phi|^2 \\ &= \min_{\phi \in V} \max_{\mathbf{j} \in W} \int_\Omega (\nabla\phi \cdot \mathbf{j} - \frac{1}{2}\mathbf{j}^2) \\ &= \max_{\mathbf{j} \in W} \min_{\phi \in V} \int_\Omega (\nabla\phi \cdot \mathbf{j} - \frac{1}{2}\mathbf{j}^2) \end{aligned} \tag{A.17}$$

The last equality of (A.17) is true because $F[\phi]$ satisfies the conditions of Proposition

5.2 in Chapter 3 [18], then we can change the order of min and max. So we have

$$\begin{aligned}
\mathcal{E} &= \max_{\mathbf{j}\in W}\min_{\phi\in V}\int_\Omega(\nabla\phi\cdot\mathbf{j}-\frac{1}{2}\mathbf{j}^2)\\
&= \max_{\mathbf{j}\in W}\left\{-\frac{1}{2}\int_\Omega\mathbf{j}^2+\min_{\phi\in V}\int_\Omega\nabla\phi\cdot\mathbf{j}\right\}\\
&= \max_{\mathbf{j}\in W}\left\{-\frac{1}{2}\int_\Omega\mathbf{j}^2+\min_{\phi\in V}\left[-\int_\Omega\nabla\cdot\mathbf{j}\phi+\int_{\Gamma_D}\phi\mathbf{j}\cdot\mathbf{n}+\int_{\Gamma_N}\phi\mathbf{j}\cdot\mathbf{n}+\sum_{i\in\mathcal{S}}\int_{\partial D_i}\phi\mathbf{j}\cdot\mathbf{n}\right]\right\}
\end{aligned}\tag{A.18}$$

Considering the conditions for $\mathbf{j}\in W$, we will have

$$\begin{aligned}
\mathcal{E} &= \max_{\mathbf{j}\in W}\left\{-\frac{1}{2}\int_\Omega\mathbf{j}^2+\min_{\phi\in V}\int_{\Gamma_D}\psi\mathbf{j}\cdot\mathbf{n}\right\}\\
&= \max_{\mathbf{j}\in W}\left\{-\frac{1}{2}\int_\Omega\mathbf{j}^2+\int_{\Gamma_D}\psi\mathbf{j}\cdot\mathbf{n}\right\}
\end{aligned}\tag{A.19}$$

So the dual problem of (A.2) is the maximization problem

$$\mathcal{E} = \max_{\mathbf{j}\in W}\left\{\int_{\Gamma_D}\psi\mathbf{j}\cdot\mathbf{n}-\frac{1}{2}\int_\Omega\mathbf{j}^2\right\}\tag{A.20}$$

with the space W given in (A.15).

A.4 Functions in H^1

In this thesis, we need to construct trial functions in $H^1(\Omega)$ where $\Omega\in\mathbb{R}^2$. However, we need to construct the function piece by piece sometimes, and the following lemma will be useful.

Lemma A.4.1. *Suppose $\Omega=\Omega_1\cup\Omega_2\cup\gamma$ is a Lipschitz domain in $\mathbb{R}^2$, where $\gamma=\partial\Omega_1\cap\partial\Omega_2$ is the interface shared by the Lipschitz domains Ω_1 and Ω_2. Suppose*

$u_i \in H^1(\Omega_i)(i = 1, 2)$, and

$$||\gamma_0 u_1 - \gamma_0 u_2||_{L^2(\gamma)} = 0,$$

where $\gamma_0 u_i$ is the trace of $u_i(i = 1, 2)$ on γ. Then

$$u = \begin{cases} \gamma_0 u_1, & \textit{on } \gamma \\ u_1, & \textit{in } \Omega_1 \\ u_2, & \textit{in } \Omega_2 \end{cases} \tag{A.21}$$

belongs to $H^1(\Omega)$.

Proof. Let

$$\Gamma_1 = \partial\Omega \cap \partial\Omega_1 \text{ and } \Gamma_2 = \partial\Omega \cap \partial\Omega_2.$$

Then $\partial\Omega = \Gamma_1 \cup \Gamma_2$. Suppose $\mathbf{n}_i(i = 1, 2)$ is the outward unit normal of the domain Ω_i. Notice that $\mathbf{n}_1 = -\mathbf{n}_2$ on γ.

Define

$$\nabla u = \begin{cases} \nabla u_1, & \text{in } \Omega_1 \\ \nabla u_2, & \text{in } \Omega_2 \end{cases} \tag{A.22}$$

Because γ is a measure zero set in Ω, we can give any reasonable definition for ∇u on γ. Also because $u_i \in H^1(\Omega_i)(i = 1, 2)$, we have $\nabla u \in L^2(\Omega)$. The left thing is to prove ∇u is the weak derivative of u in Ω.

For any vector function $\mathbf{v} \in (C_0^\infty(\Omega))^2$,

$$|\int_\gamma (u_1 - u_2)\mathbf{v}\cdot\mathbf{n}_1| \le ||\gamma_0 u_1 - \gamma_0 u_2||_{L^2(\gamma)} \cdot ||\mathbf{v}\cdot\mathbf{n}_1||_{L^2(\gamma)} = 0$$

So we have $\int_\gamma (u_1 - u_2)\mathbf{v}\cdot\mathbf{n}_1 = 0$.

From Green's identities

$$\begin{aligned}
-\int_\Omega u\nabla\cdot\mathbf{v} &= -\int_{\Omega_1} u_1\nabla\cdot\mathbf{v} - \int_{\Omega_2} u_2\nabla\cdot\mathbf{v} \\
&= \int_{\Omega_1} \nabla u_1\cdot\mathbf{v} - \int_{\Gamma_1} u_1\mathbf{v}\cdot\mathbf{n}_1 - \int_\gamma u_1\mathbf{v}\cdot\mathbf{n}_1 \\
&+ \int_{\Omega_2} \nabla u_2\cdot\mathbf{v} - \int_{\Gamma_2} u_2\mathbf{v}\cdot\mathbf{n}_2 - \int_\gamma u_2\mathbf{v}\cdot\mathbf{n}_2 \\
&= \int_\Omega \nabla u\cdot\mathbf{v} - \int_\gamma (u_1 - u_2)\mathbf{v}\cdot\mathbf{n}_1 \\
&= \int_\Omega \nabla u\cdot\mathbf{v}.
\end{aligned} \tag{A.23}$$

Which means ∇u is the weak derivative of u in the whole space Ω, so $u \in H^1(\Omega)$. $\square$

In order to construct a function in $H^1(\Omega)$, we can construct it piece by piece in different subdomains and let them matched each other on the interface.

A.5 The polar coordinate system

Let $(\mathbf{e}_x, \mathbf{e}_y)$ be basis vectors of Cartesian coordinate system (x, y), and $(\mathbf{u}_r, \mathbf{u}_\theta)$ be basis vectors of the polar coordinate system (r, θ). We have

$$\begin{aligned}
\mathbf{e}_x &= \cos\theta\mathbf{u}_r - \sin\theta\mathbf{u}_\theta, \\
\mathbf{e}_y &= \sin\theta\mathbf{u}_r + \cos\theta\mathbf{u}_\theta,
\end{aligned} \tag{A.24}$$

For a function $\phi(r,\theta)$,

$$\begin{aligned}\nabla\phi(r,\theta) &= \frac{\partial\phi}{\partial r}\mathbf{u}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\mathbf{u}_\theta,\\ \nabla^\perp\phi(r,\theta) &= -\frac{1}{r}\frac{\partial\phi}{\partial\theta}\mathbf{u}_r + \frac{\partial\phi}{\partial r}\mathbf{u}_\theta,\end{aligned} \tag{A.25}$$

For a vector function $\mathbf{j} = j_r\mathbf{u}_r + j_\theta\mathbf{u}_\theta$,

$$\nabla\cdot\mathbf{j} = \frac{1}{r}\frac{\partial}{\partial r}(rj_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(j_\theta). \tag{A.26}$$

Then

$$\Delta\phi(r,\theta) = \nabla\cdot(\nabla\phi) = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \tag{A.27}$$

Appendix B

Local approximation

B.1 Properties of some functions

In this thesis, we need to use some properties of some functions. The proof of these properties are very easy and we will state them in this section.

Proposition B.1.1. *When $a \in (0,1)$ and $b \geq 1$, the following function*

$$f(k) = \frac{ka^{bk}}{1 - a^{bk}}$$

is a monotonically decreasing function for $k \geq 1$.

The proof for this proposition is just basic calculus. Notice that for any positive integer k, we have

$$f(k) \leq f(1) = \frac{a^b}{1 - a^b}.$$

B.2 The approximation in Δ_{ij}

We need to show that we can have an extension of the trial function ϕ from $\Pi_B \cup B_0$ into the domain Δ, actually fro $\Pi_{ij} \cup B_{ij}$ into the domain Δ_{ij} locally.

In order to use the Kirszbraun's theorem introduced in the lemma 3.2.2. We need to evaluate $|\nabla\phi|$ near $\partial\Delta_{ij}$ in $\Pi_{ij} \cup B_{ij}$.

In the neck Π_{ij}, we already show in the section 3.2 that

$$|\nabla\phi| \leq \frac{C}{R}(\mathcal{U}_i - \mathcal{U}_j)^2 \qquad \text{on } \partial\Delta_{ij} \cap \partial\Pi_{ij}$$

In the domain B_{ij}, the trial function is given in (4.8). The weight functions w_k, w only depends on θ because $d(\theta) \equiv R/2$. On the boundary $\partial\Delta_{ij} \cap \partial B_{ij}$

$$\begin{aligned} |\nabla\phi|^2 &= \left|\frac{\partial w_k}{\partial r}\cos k\theta + \frac{\partial w}{\partial r}\mathcal{L}(\mathcal{U})\right|^2_{r=L-R/2} + \left|-\frac{k}{r}w_k \sin k\theta + \frac{1}{r}w\frac{\partial \mathcal{L}(\mathcal{U})}{\partial\theta}\right|^2_{r=L-R/2} \\ &\leq C\left|\frac{k(1-R/(2L))^{2k}}{1-(1-R/(2L))^{2k}}\right|^2 + C\left|\frac{1}{\ln(1-R/(2L))}\right|^2 + C\left|\frac{\mathcal{U}_i - \mathcal{U}_j}{\alpha_{ij}}\right|^2 \\ &\leq \frac{C}{R^2} + \frac{C}{R^2}(\mathcal{U}_i - \mathcal{U}_j)^2 \leq \frac{C}{R^2} \end{aligned}$$

where C is a constant and $\alpha_{ij} = O(R)$ is defined in (4.45). Here we also used the proposition B.1.1 and the fact that

$$\left|\frac{1}{\ln(1-R/(2L))}\right| = \frac{2L}{R} + O(1).$$

From the Kirszbraun's theorem, we can extend the trial function from $\Pi_{ij} \cup B_{ij}$ into the domain Δ_{ij}, and it will satisfy

$$|\nabla\phi| \leq C/R \qquad \text{in } \Delta_{ij} \tag{B.1}$$

It is easy to show that the area of Δ_{ij} is $O(R^2)$. Hence we have

$$\int_{\Delta_{ij}} |\nabla\phi|^2 \leq \frac{C}{R^2}|\Delta_{ij}| = O(1). \tag{B.2}$$

B.3 Some properties of $d(\theta)$ in B_i

Without loss of generality, we can suppose that $\theta_i = 0$. This assumption will not affect the approximation for $G_i(\theta)$ in (4.40).

Now we have

$$\begin{aligned} d(\theta) &= L - r_i\cos(\theta) - \sqrt{R^2 - (r_i \sin\theta)^2} \\ &= \delta_i + r_i(1-\cos(\theta)) + (R - \sqrt{R^2 - (r_i\sin(\theta))^2}) \\ &> R - \sqrt{R^2 - (r_i\sin(\theta))^2} \end{aligned} \tag{B.3}$$

and $d(\theta)$ is bounded above by $R/2$.

Also we have

$$\begin{aligned} d'(\theta) &= r_i\sin(\theta)\left(1 + \frac{r_i\cos\theta}{\sqrt{R^2-(r_i\sin\theta)^2}}\right) \\ &= \frac{r_i\sin(\theta)}{\sqrt{R^2-(r_i\sin\theta)^2}}(L - d(\theta) \end{aligned} \tag{B.4}$$

Considering the area the triangle $OO_iP_i^+$ and using Heron's formula, there will be

$$\frac{1}{2}\rho r_i \sin\alpha_i = \sqrt{\frac{(r_i+\rho+R)(-r_i+\rho+R)(r_i-\rho+R)(r_i+\rho-R)}{16}}$$

From (2.41) and (2.42), we have $r_i \approx L - R$, also notice that $\rho = L - R/2$. Hence

$$
\begin{aligned}
r_i \sin\alpha_i &\approx \frac{1}{2(L-R/2)}\sqrt{(2L-R/2)(3R/2)(R/2)(2L-5R/2)} \\
&= \frac{\sqrt{3}R}{2(L-R/2)}\sqrt{(L-R/4)(L-5R/4)} \\
&\leq \frac{\sqrt{3}R}{2(L-R/2)}\sqrt{(L-3R/4)^2} \\
&= \frac{\sqrt{3}R(L-3R/4)}{2(L-R/2)} \\
&\leq \frac{\sqrt{3}R}{2}
\end{aligned} \tag{B.5}
$$

From(B.5), we have for any $\theta \in (-\alpha_i, \alpha_i)$

$$
r_i \sin(\theta) \leq r_i \sin(\alpha_i) \leq \frac{\sqrt{3}R}{2} \tag{B.6}
$$

So we have

$$
\begin{aligned}
|d'(\theta)| &= \frac{|r_i \sin\theta|}{\sqrt{R^2 - (r_i \sin\theta)^2}}(L - d(\theta)) \\
&\leq \frac{L|r_i \sin\theta|}{\sqrt{R^2 - 3R^2/4}} = \frac{2L}{R}|r_i \sin\theta| \\
&\leq \sqrt{3}L = O(1)
\end{aligned} \tag{B.7}
$$

and

$$
\begin{aligned}
\frac{d'^2(\theta)}{d(\theta)} &\leq \frac{4}{R^2}\frac{(r_i \sin\theta)^2}{R - \sqrt{R^2 - (r_i \sin\theta)^2}} \\
&= \frac{4L^2}{R^2}(R + \sqrt{R^2 - (r_i \sin\theta)^2}) \\
&\leq \frac{4L^2}{R^2}(R + R) = \frac{8L^2}{R} = O(L^2 R^{-1})
\end{aligned} \tag{B.8}
$$

Also because $r_i \cos\theta \le L$ and $r_i \sin\theta \le \sqrt{3}R/2$, we have

$$\begin{aligned} |d''(\theta)| &= \left| \frac{r_i \cos\theta R^2}{(R^2 - (r_i \sin\theta)^2)^{3/2}} - \frac{(r_i \sin\theta)^2}{R^2 - (r_i \sin\theta)^2} \right| (L - d(\theta)) \\ &\le \max\{\frac{8L}{R} r_i \cos\theta, 3L\} \\ &\le \frac{8L^2}{R} = O(L^2 R^{-1}) \end{aligned} \tag{B.9}$$

Since $\delta_i \le d(\theta) \le R/2 \ll L$ we have following approximation

$$-\frac{1}{\ln(1 - d/L)} = \frac{L}{d} + O(1).$$

Here we will talk about the approximation for integration of $1/d(\theta)$ in B_i. Remember that

$$d(\theta) = \delta_i + r_i(1 - \cos(\theta)) + R(1 - \sqrt{1 - (r_i \sin(\theta)/R)^2})$$

where $\theta \in (-\alpha_i, \alpha_i)$ with $\alpha_i = O(R)$.

First from Taylor's expansion theorem, we can easily have

$$|r_i(1 - \cos(\theta)) - \frac{r_i \theta^2}{2}| \le \frac{r_i \theta^4}{24} \tag{B.10}$$

Also from Taylor expansion theorem, we can easily have

$$\frac{1}{2}x \le 1 - \sqrt{1 - x} \le \frac{1}{2}x + \frac{x^2}{8(1 - \gamma)^{3/2}} \qquad \text{for all } x \in [0, \gamma] \tag{B.11}$$

where $0 < \gamma < 1$ is a positive constant.

Notice that we already proved (B.6). From the (B.11), we will have

$$\begin{aligned}
&\left|(1-\sqrt{1-(r_i\sin(\theta)/R)^2})-\frac{1}{2}(\frac{r_i\theta}{R})^2\right| \\
&\leq \left|(1-\sqrt{1-(r_i\sin(\theta)/R)^2})-\frac{1}{2}(\frac{r_i\sin\theta}{R})^2\right|+\left|\frac{1}{2}(\frac{r_i\sin\theta}{R})^2-\frac{1}{2}(\frac{r_i\theta}{R})^2\right| \\
&\leq C(\frac{r_i\sin(\theta)}{R})^4+C(\frac{r_i}{R})^2\theta^4 \\
&\leq C\left(\frac{L\theta}{R}\right)^4
\end{aligned} \tag{B.12}$$

where C is a constant which does not depend on δ_i or R.

Denota the approximation for $d(\theta)$ as

$$\tilde{d}(\theta)=\delta_i+\frac{r_iL}{2R}\theta^2=\delta_i+\frac{r_i\theta^2}{2}+\frac{R}{2}(\frac{r_i\theta}{R})^2+\frac{r_i\delta_i}{2R}\theta^2.$$

Put above bounds together, we will have

$$\begin{aligned}
&|d(\theta)-\tilde{d}(\theta)|=\left|d(\theta)-\left(\delta_i+\frac{r_i\theta^2}{2}+\frac{R}{2}(\frac{r_i\theta}{R})^2+\frac{r_i\delta_i}{2R}\theta^2\right)\right| \\
&\leq \left|r_i(1-\cos(\theta))-\frac{r_i\theta^2}{2}\right|+R\left|(1-\sqrt{1-(r_i\sin(\theta)/R)^2})-\frac{1}{2}(\frac{r_i\theta}{R})^2\right|+\left|\frac{r_i\delta_i}{2R}\theta^2\right| \\
&\leq \frac{r_i\theta^4}{24}+C(\frac{L\theta}{R})^4R+CL\delta_iR^{-1}\theta^2 \\
&\leq CL^4R^{-3}\theta^4+CL\delta_iR^{-1}\theta^2
\end{aligned} \tag{B.13}$$

Next we will prove that

$$\left|\frac{1}{d(\theta)}-\frac{1}{\tilde{d}(\theta)}\right|\leq O(R^{-1}) \qquad \text{for all } \theta\in(-\alpha_i,\alpha_i) \tag{B.14}$$

We need to prove the above bound in different regions. We first divide the region

$(-\alpha_i, \alpha_i)$ into two regions

$$|\theta| \le \beta_i \text{ and } \beta_i \le |\theta| \le \alpha_i$$

where $\beta_i = \sqrt{2R\delta_i}/L$. Actually, any $\beta_i = O(\sqrt{R\delta_i}/L)$ would be fine to get the bound (B.14).

When $|\theta| \le \beta_i$, we have

$$\begin{aligned} &\left| d(\theta) - \tilde{d}(\theta) \right| \le CR^{-3}\theta^4 + CL\delta_i R^{-1}\theta^2 \le CR^{-1}\delta_i^2 \\ &d(\theta) \ge \delta_i \text{ and } \tilde{d}(\theta) \ge \delta_i \end{aligned} \tag{B.15}$$

hence we have

$$\left| \frac{1}{d(\theta)} - \frac{1}{\tilde{d}(\theta)} \right| = \left| \frac{d(\theta) - \tilde{d}(\theta)}{d(\theta)\tilde{d}(\theta)} \right| \le \frac{CR^{-1}\delta_i^2}{\delta_i^2} = O(R^{-1}).$$

When $\beta_i \le |\theta| \le \alpha_i$, first we will easily have

$$\tilde{d}(\theta) = \delta_i + \frac{r_i L}{2R}\theta^2 \ge \frac{r_i L}{2R}\theta^2 = CR^{-1}L^2\theta^2$$

Using the formula (B.11), we have

$$d(\theta) \ge R(1 - \sqrt{1 - (r_i \sin\theta/R)^2}) \ge \frac{r_i^2}{2R}(\sin\theta)^2 \ge \frac{r_i^2}{2R}(\frac{2}{\pi}\theta)^2 \ge CR^{-1}L^2\theta^2,$$

here we used the formula $\sin\theta \ge \frac{2}{\pi}\theta \quad \forall\theta \in [0, \pi/2]$.

Hence in the region $\beta_i \le |\theta| \le \alpha_i$, we also have

$$\left| \frac{1}{d(\theta)} - \frac{1}{\tilde{d}(\theta)} \right| = \left| \frac{d(\theta) - \tilde{d}(\theta)}{d(\theta)\tilde{d}(\theta)} \right| \le \frac{CR^{-3}L^4\theta^4 + CL\delta_i R^{-1}\theta^2}{(CR^{-1}L^2\theta^2)^2} \le O(R^{-1}).$$

In the summary, we proved (B.14).

Notice that $\alpha_i = O(R/L)$, we have

$$\begin{aligned}\int_{-\alpha_i}^{\alpha_i} \frac{Ld\theta}{d(\theta)} = \int_{-\alpha_i}^{\alpha_i} \frac{Ld\theta}{\tilde{d}(\theta)} + O(1) &= \int_{-\alpha_i}^{\alpha_i} \frac{Ld\theta}{\delta_i + \frac{Lr_i}{2R}\theta^2} + O(1) \\ &= 2\sqrt{\frac{2LR}{r_i\delta_i}} \arctan \sqrt{\frac{Lr_i}{2R\delta_i}}\alpha_i + O(1) \\ &= 2\sqrt{\frac{2LR}{r_i\delta_i}}(\frac{\pi}{2} - \frac{\sqrt{2R\delta_i}}{\sqrt{Lr_i}\alpha_i}) + O(1) \\ &= \sqrt{\frac{2LR}{r_i\delta_i}}\pi + O(1)\end{aligned} \tag{B.16}$$

Notice that

$$\sqrt{\frac{Lr_i}{2R\delta_i}}\alpha_i = O(\sqrt{\frac{R}{\delta_i}}) \gg 1,$$

and we used the following approximation above

$$\arctan(x) = \frac{\pi}{2} - \frac{1}{x} + O(x^{-3}), \quad \text{as } x \to \infty.$$

B.4 The approximation for $G_i(\theta)$

In order to approximate $G_i(\theta)$ in (4.40), we need to approximate the following three integrals one by one

$$\begin{aligned} G_{i1}^k(\theta) &:= \int_{L-d(\theta)}^{L} \frac{dr}{r} \left(\int_r^L \frac{ds}{s} (-2k \frac{\partial w_k(s,\theta)}{\partial \theta}) \right)^2 \\ G_{i2}^k(\theta) &:= \int_{L-d(\theta)}^{L} \frac{dr}{r} \left(\int_r^L \frac{ds}{s} \frac{\partial^2 w_k(s,\theta)}{\partial \theta^2} \right)^2 \\ G_{i3}(\theta) &:= \int_{L-d(\theta)}^{L} \frac{dr}{r} \left(\int_r^L \frac{ds}{s} \frac{\partial^2 w(s,\theta)}{\partial \theta^2} \right)^2 \end{aligned} \tag{B.17}$$

In this situation, we also denote $p(\theta) = (1 - d(\theta)/L)^k$ for simplicity. Then from (4.9)

$$
\begin{aligned}
w_k(r,\theta) &= \frac{(r/L)^k - p^2(r/L)^{-k}}{1-p^2} = \frac{(r/L)^k - (r/L)^{-k}}{1-p^2} + (r/L)^{-k} \\
\frac{\partial w_k(r,\theta)}{\partial \theta} &= [(r/L)^k - (r/L)^{-k}]\left(\frac{-2kp^2 d'}{(1-p^2)^2(L-d)}\right) \\
\frac{\partial^2 w_k(r,\theta)}{\partial \theta^2} &= k[(r/L)^k - (r/L)^{-k}]\left(\frac{p^2((4k+2)p^2+4k-2)d'^2}{(1-p^2)^3(L-d)^2} + \frac{-2p^2 d''}{(1-p^2)^2(L-d)}\right)
\end{aligned}
\tag{B.18}
$$

and

$$
\begin{aligned}
w(r,\theta) &= \frac{1}{\ln(1-d/L)} \ln\frac{r}{L} \\
\frac{\partial w(r,\theta)}{\partial \theta} &= \frac{d'}{(L-d)(\ln(1-d/L))^2} \ln(\frac{r}{L}) \\
\frac{\partial^2 w(r,\theta)}{\partial \theta^2} &= \left(\frac{2d'^2}{(L-d)^2(\ln(1-d/L))^3} + \frac{d'^2 + (L-d)d''}{(L-d)^2(\ln(1-d/L))^2}\right) \ln(\frac{r}{L})
\end{aligned}
\tag{B.19}
$$

We will use C to denote some general constant which will change in different cases,but doesn't depend on k, R or δ in the following approximation.

1. Because $\delta \le d(\theta) \le R/2$, we have

$$(1-d/L)^{2k-1} \le Cp^2 \text{ and } (1-d/L)^{2k-2} \le Cp^2$$

and

$$(2-(r/L)^k - (r/L)^{-k})^2 \le (2-(1-d/L)^k - (1-d/L)^{-k})^2 = p^{-2}(1-p)^4, \quad \forall r \in (L-d, L)$$

also

$$\frac{1}{1-p^2} = \frac{1}{(1-p)(1+p)} \le \frac{1}{1-p}$$

Then we have

$$\begin{aligned}
G_{i1}^k(\theta) &= \int_{L-d(\theta)}^{L} \frac{dr}{r} \left(\int_r^L \frac{ds}{s} (-2k \frac{\partial w_k(s,\theta)}{\partial \theta}) \right)^2 \\
&\le C \left(\frac{kp^2 d'}{(L-d)(1-p)^2} \right)^2 \int_{L-d(\theta)}^{L} \frac{dr}{r} \left(\int_{r/L}^1 \frac{kdt}{t} (t^k - t^{-k}) \right)^2 \\
&= C \left(\frac{kp^2 d'}{(L-d)(1-p)^2} \right)^2 \int_{L-d(\theta)}^{L} \frac{dr}{r} (2 - (r/L)^k - (r/L)^{-k})^2 \qquad \text{(B.20)} \\
&\le C \left(\frac{kd'p^2}{(L-d)(1-p)^2} \right)^2 \int_{L-d(\theta)}^{L} \frac{dr}{r} p^{-2}(1-p)^4 \\
&= C(d'/L)^2 k^2 p^2 [-\ln(1-d/L)]
\end{aligned}$$

where we changed parameter $t = s/L$ on the second line.

Here $k^2p^2 = k^2(1-d/L)^{2k}$ will get it's maximal value when $k = -1/\ln(1-d/L)$, which means

$$k^2p^2 \le \frac{1}{[-\ln(1-d/L)]^2} e^{-2}$$

Then we have

$$G_{i1}(\theta) \le C \frac{d'^2}{L^2[-\ln(1-d/L)]} \le C \frac{d'^2}{Ld} \le C \frac{L}{R} \qquad \text{(B.21)}$$

because $d'^2/d = O(L^2R^{-1})$.

2. Similarly, by integration over s we will have

$$\begin{aligned}
G_{i2}^k(\theta) &= \int_{L-d(\theta)}^{L} \frac{dr}{r} \left(\int_r^L \frac{ds}{s} \frac{\partial^2 w_k(s,\theta)}{\partial \theta^2} \right)^2 \\
&\le C \left(\frac{d'^2 p^2 [(4k+2)p^2 + (4k-2)]}{(L-d)^2(1-p^2)^3} \right)^2 [-\ln(1-d/L)] p^{-2}(1-p)^4 \qquad \text{(B.22)} \\
&+ C \left(\frac{-2d'' p^2}{(L-d)(1-p^2)^2} \right)^2 [-\ln(1-d/L)] p^{-2}(1-p)^4
\end{aligned}$$

Since $(4k+2)p^2 + (4k-2) \le (4k+2) + (4k-2) = 8k$,

$$\begin{aligned} G_{i2}^k(\theta) &\le C\frac{d'^4}{(L-d)^4}\frac{k^2p^2}{(1-p)^2}[-\ln(1-d/L)] + C(d''/L)^2p^2[-\ln(1-d/L)] \\ &\le C\frac{d'^4}{(L-d)^4}\frac{k^2p^2}{(1-p)^2}\frac{d}{L} + C(LR^{-1})^2\frac{d}{L} \end{aligned} \tag{B.23}$$

From the Proposition B.1.1, we have

$$\frac{k^2p^2}{(1-p)^2} = \left(\frac{k(1-d/L)^k}{1-(1-d/L)^k}\right)^2 \le \left(\frac{(1-d/L)}{1-(1-d/L)}\right)^2 \le C(\frac{L}{d})^2$$

Then we have

$$G_{i2}^k(\theta) \le C\frac{d'^4}{L^3d} + C\frac{L}{R} \le C\frac{d'^2}{Ld} + C\frac{L}{R} \le C\frac{L}{R}, \tag{B.24}$$

because $d'/L = O(1)$ and $d'^2/d = O(L^2R^{-1})$.

3. For the last integration,

$$\begin{aligned} G_{i3}(\theta) &= \int_{L-d(\theta)}^{L}\frac{dr}{r}\left(\int_r^L\frac{ds}{s}\frac{\partial^2 w(s,\theta)}{\partial\theta^2}\right)^2 \\ &= \left(\frac{2d'^2}{(L-d)^2[\ln(1-d/L)]^3} + \frac{d'^2 + d''(L-d)}{(L-d)^2[\ln(1-d/L)]^2}\right)^2\int_{L-d(\theta)}^{L}\frac{dr}{r}(\int_{r/L}^{1}\frac{\ln t dt}{t})^2 \\ &= \left(\frac{[2+\ln(1-d/L)]d'^2}{[\ln(1-d/L)]^3(L-d)^2} + \frac{d''}{(L-d)[\ln(1-d/L)]^2}\right)^2\frac{-(\ln(1-d/L))^5}{20} \\ &\le C\frac{d'^4}{-\ln(1-d/L)L^4} + C\frac{d''^2}{L^2}[-\ln(1-d/L)] \\ &\le C\frac{d'^4}{L^3d} + C(\frac{L}{R})^2\frac{d}{L} \le C\frac{d'^2}{Ld} + C(\frac{L}{R})^2\frac{R}{L} \\ &\le C\frac{L}{R} \end{aligned} \tag{B.25}$$

In conclusion, we have

$$\frac{1}{2}\int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} G_i(\theta)d\theta = \frac{1}{2}\int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \left(G_{i1}^k(\theta) + G_{i2}^k(\theta) + G_{i3}(\theta)\right) d\theta \leq C\frac{L\alpha_i}{R} = O(1) \quad \text{(B.26)}$$

because $\alpha_i = O(R/L)$.

B.5 The approximation for a_i, b_i, c_i

In this section, we will approximate a_i, b_i and c_i separately.

Approximation for a_i

First let's look at a_i,

$$\begin{aligned} a_i &= \int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} d\theta \int_{L-d(\theta)}^{L} rdr \left((\frac{\partial w}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial w}{\partial \theta})^2 \right) \\ &= \int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \left(\frac{1}{-\ln(1-d/L)} + \frac{d'^2}{3(L-d)^2(-\ln(1-d/L))} \right) d\theta \end{aligned} \quad \text{(B.27)}$$

We can bound the second integration like following

$$\left| \int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{d'^2}{3(L-d)^2(-\ln(1-d/L))} d\theta \right| \leq C \left| \int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{d'^2}{Ld} d\theta \right| \leq C \left| \int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{L}{R} d\theta \right| = O(1)$$

Also notice that

$$\frac{1}{-\ln(1-d/L)} = \frac{L}{d} + O(1),$$

In this approximation, we requires that $R \ll L$.

Hence from (B.16), we will have

$$\begin{aligned} a_i &= \int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{d\theta}{-\ln(1-d(\theta)/L)} + O(1) = \int_{-\alpha_i}^{\alpha_i} \frac{Ldt}{d(\theta_i+t)} + O(1) \\ &= \pi\sqrt{\frac{2LR}{r_i\delta_i}} + O(1) \end{aligned} \tag{B.28}$$

where

$$d(\theta_i+t) = L - r_i\cos t - \sqrt{R^2-(r_i\sin t)^2} = \delta_i + r_i(1-\cos t) + (R - \sqrt{R^2-(r_i\sin t)^2})$$

for $t \in (-\alpha_i, \alpha_i)$. Here we changed parameter $\theta = \theta_i + t$.

Approximation for b_i

For b_i, we have

$$\begin{aligned} b_i &= \int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} d\theta \int_{L-d(\theta)}^{L} rdr \left\{ (\frac{\partial w_k}{\partial r}\cos k\theta)\frac{\partial w}{\partial r} + \frac{1}{r^2}(-kw_k\sin k\theta + \frac{\partial w_k}{\partial \theta}\cos k\theta)\frac{\partial w}{\partial \theta} \right\} \\ &= \int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{\cos k\theta}{\ln(1-d/L)} d\theta + \int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{kd'\sin k\theta}{(L-d)} \frac{1-p^2+2p\ln p}{(\ln p)^2(1-p^2)} d\theta \\ &- \int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{2d'^2\cos k\theta}{(L-d)^2\ln(1-d/L)} \frac{p[1-p^2+(1+p^2)\ln p]}{\ln p(1-p^2)^2} d\theta \end{aligned} \tag{B.29}$$

where $p(\theta) = (1-d(\theta)/L)^k$.

We can first show that

$$\begin{aligned} F_1(p) &= \frac{1-p^2+2p\ln p}{(\ln p)^2(1-p^2)} \\ F_2(p) &= \frac{p[1-p^2+(1+p^2)\ln p]}{\ln p(1-p^2)^2} \\ F_3(p) &= \frac{(1-p^2)^2+(1-p^2)p\ln p-(1+p^2)p(\ln p)^2}{(\ln p)^2(1-p^2)^2} \end{aligned}$$

are bounded for any $p \in (0,1)$. This is easy to see because the functions of p above are bounded when $p \to 0$ or $p \to 1$.

Also from B.3, we have

$$\frac{d'}{L-d} = O(1), \frac{d'^2}{d} = O(L^2R^{-1}) \text{ and } d'' = O(L^2R^{-1})$$

From integration by parts and $\alpha_i = O(R/L)$, we will have

$$\begin{aligned}
&\left|\int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{kd'\sin k\theta}{(L-d)} \frac{1-p^2+2p\ln p}{(\ln p)^2(1-p^2)} d\theta\right| = \left|\int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{d'}{(L-d)} \frac{1-p^2+2p\ln p}{(\ln p)^2(1-p^2)} d(\cos k\theta)\right| \\
&= O(1) + \left|\int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \cos k\theta d\left\{\frac{d'}{(L-d)} \frac{1-p^2+2p\ln p}{(\ln p)^2(1-p^2)}\right\}\right| \\
&\leq O(1) + \left|\int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{(d''(L-d)+d'^2)\cos k\theta}{(L-d)^2} \frac{1-p^2+2p\ln p}{(\ln p)^2(1-p^2)} d\theta\right| \\
&+ \left|\int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{2d'^2\cos k\theta}{(L-d)^2\ln(1-d/L)} \frac{(1-p^2)^2+(1-p^2)p\ln p-(1+p^2)p(\ln p)^2}{(\ln p)^2(1-p^2)^2} d\theta\right| \\
&\leq O(1) + C\left|\int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{d''(L-d)+d'^2}{(L-d)^2} d\theta\right| + C\left|\int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{2d'^2}{(L-d)^2\ln(1-d/L)}\right| \\
&\leq O(1) + O(LR^{-1})\alpha_i + O(LR^{-1})\alpha_i \\
&= O(1)
\end{aligned}$$

We also have

$$\begin{aligned}
&\left|\int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{2d'^2\cos k\theta}{(L-d)^2\ln(1-d/L)} \frac{p[1-p^2+(1+p^2)\ln p]}{\ln p(1-p^2)^2} d\theta\right| \\
&\leq C\left|\int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{d'^2}{Ld} d\theta\right| \leq O(LR^{-1})\alpha_i = O(1).
\end{aligned}$$

So we have

$$b_i = \int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{\cos k\theta}{\ln(1-d(\theta)/L)} d\theta + O(1). \tag{B.30}$$

Also from the approximation (B.16), we will have

$$\int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{\cos k\theta}{\ln(1-d(\theta)/L)} d\theta = -\int_{-\alpha_i}^{\alpha_i} \frac{L\cos k(\theta_i+t)}{\delta_i + \frac{Lr_i t^2}{2R}} dt + O(1).$$

What we need to approximate is

$$\begin{aligned} &\int_{-\alpha_i}^{\alpha_i} \frac{L\cos k(\theta_i+t)}{\delta_i + \frac{Lr_i t^2}{2R}} dt = \Re\left\{ L\int_{-\alpha_i}^{\alpha_i} \frac{e^{ik(\theta_i+t)}}{\delta_i + \frac{r_i t^2}{2R}} dt \right\} \\ &= \Re\left\{ Le^{ik\theta_i} \int_{-\alpha_i}^{\alpha_i} \frac{e^{ikt}}{\delta_i + \frac{Lr_i t^2}{2R}} dt \right\} = \Re\left\{ \sqrt{\frac{2LR}{r_i\delta_i}} e^{ik\theta_i} \int_{-S_i}^{S_i} \frac{e^{ik\sqrt{2R\delta_i/(Lr_i)}x}}{1+x^2} dx \right\} \end{aligned} \tag{B.31}$$

Here we changed parameter and

$$S_i = \frac{\alpha_i}{\sqrt{2R\delta_i/(Lr_i)}} = O(\sqrt{\frac{R}{\delta_i}}) \gg 1.$$

From the residue formula, we have

$$\int_{-S_i}^{S_i} \frac{e^{ik\sqrt{2R\delta_i/(Lr_i)}x}}{1+x^2} dx + \int_{C_{S_i}} \frac{e^{ik\sqrt{2R\delta_i/(Lr_i)}x}}{1+x^2} dx = \pi e^{-k\sqrt{2R\delta_i/(Lr_i)}} \tag{B.32}$$

where $C_{S_i} = \{S_i e^{i\theta}, 0 \le \theta \le \pi\}$ is the circle with radius S_i in the upper half complex plane. And

$$\left| \int_{C_{S_i}} \frac{e^{ik\sqrt{2R\delta_i/(Lr_i)}x}}{1+x^2} dx \right| \le \left| \int_{C_{S_i}} \frac{1}{1+S_i^2 e^{2i\theta}} dx \right| \le \frac{\pi S_i}{S_i^2 - 1} = O(S_i^{-1}) = O(\sqrt{\frac{\delta_i}{R}}).$$

Hence we have

$$\int_{-S_i}^{S_i} \frac{e^{ik\sqrt{2R\delta_i/(Lr_i)}x}}{1+x^2} dx = \pi e^{-k\sqrt{2R\delta_i/(Lr_i)}} + O(\sqrt{\frac{\delta_i}{R}}).$$

In the summary

$$\begin{aligned} b_i &= -\Re\left\{\sqrt{\frac{2LR}{r_i\delta_i}} e^{ik\theta_i} \int_{-S_i}^{S_i} \frac{e^{ik\sqrt{2R\delta_i/(Lr_i)}x}}{1+x^2} dx\right\} + O(1) \\ &= -\Re\left\{\sqrt{\frac{2LR}{r_i\delta_i}} e^{ik\theta_i} (\pi e^{-k\sqrt{2R\delta_i/(Lr_i)}} + O(\sqrt{\frac{\delta_i}{R}}))\right\} + O(1) \\ &= -\pi\sqrt{\frac{2LR}{r_i\delta_i}} \frac{\cos k\theta_i}{e^{k\sqrt{2R\delta_i/(Lr_i)}}} + O(1) \end{aligned} \tag{B.33}$$

Approximation for c_i

At last, let's look at c_i,

$$\begin{aligned} c_i &= \int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} d\theta \int_{L-d(\theta)}^{L} r\left(\frac{\partial w_k}{\partial r}\cos k\theta\right)^2 + \frac{1}{r}\left(-kw_k \sin k\theta + \frac{\partial w_k}{\partial \theta}\cos k\theta\right)^2 dr \\ &= \int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \left[\frac{k}{2}\frac{1+p^2}{1-p^2}\right] d\theta - \int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \left[\frac{2k^2p^2\ln(1-d/L)}{(1-p^2)^2}\cos 2k\theta\right] d\theta \\ &+ \int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{2kd'p}{(L-d)} \frac{p[1-p^2+(1+p^2)\ln p]}{(1-p^2)^3} \sin 2k\theta d\theta \\ &+ \int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{2k^2d'^2p^2\ln(1-d/L)}{(L-d)^2(1-p^2)^2} \frac{1-p^4+4p^2\ln p}{\ln p(1-p^2)^2} \cos^2 k\theta d\theta \end{aligned} \tag{B.34}$$

Like before we can first show that

$$\begin{aligned} F_1(p) &= \frac{p[1-p^2+(1+p^2)\ln p]}{(1-p^2)^3} \\ F_2(p) &= \frac{1-p^4+4p^2\ln p}{\ln p(1-p^2)^2} \end{aligned}$$

are both bounded for any $p \in (0,1)$.

Then we will have

$$
\begin{aligned}
&\left|\int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{2kd'p}{(L-d)} \frac{p[1-p^2+(1+p^2)\ln p]}{(1-p^2)^3} \sin 2k\theta d\theta\right| \\
&\le C\left|\int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} 2kd'(L-d)^{k-1} d\theta\right| \le C\left|\int_{L-R/2}^{L-\delta_i} ks^{k-1} ds\right| \\
&\le C|(L-R/2)^k-(L-\delta_i)^k| = O(1)
\end{aligned}
$$

Here we changed parameter $s = L - d(\theta)$.

Like the proof in appendix B.4, we can prove

$$
\left|\frac{2k^2d'^2p^2\ln(1-d/L)}{(L-d)^2(1-p^2)^2}\right| \le C\frac{d'^2}{Ld},
$$

which will give us

$$
\begin{aligned}
&\left|\int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{2k^2d'^2p^2\ln(1-d/L)}{(L-d)^2(1-p^2)^2} \frac{1-p^4+4p^2\ln p}{\ln p(1-p^2)^2} \cos^2 k\theta d\theta\right| \\
&\le C\left|\int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{d'^2}{d} d\theta\right| = O(1)
\end{aligned}
$$

So we have

$$
\begin{aligned}
c_i &= \int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{k}{2}\frac{1+p^2}{1-p^2} d\theta - \int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{2k^2p^2\ln(1-d/L)}{(1-p^2)^2} \cos 2k\theta d\theta + O(1) \\
&= k\alpha_i + \int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \left(\frac{kp^2}{1-p^2} - \frac{2kp^2\ln p}{(1-p^2)^2}\cos 2k\theta\right) d\theta + O(1)
\end{aligned}
\tag{B.35}
$$

Later in this section, we will prove the following results as two propositions.

$$\int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{kp^2}{1-p^2} = \frac{\pi}{2}\sqrt{\frac{2LR}{r_i\delta_i}}\sqrt{\frac{2k\delta_i}{L\pi}}Li_{1/2}(e^{-2k\delta_i/L}) + O(1),$$
$$\int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{-2kp^2\ln p}{(1-p^2)^2}\cos 2k\theta d\theta = \frac{\pi}{2}\sqrt{\frac{2LR}{r_i\delta_i}}\exp\left[-2k\sqrt{\frac{2\delta_i R}{r_i L}}\right]\cos 2k\theta_i + O(1) \tag{B.36}$$
$$= \pi\sqrt{\frac{2LR}{r_i\delta_i}}\exp\left[-2k\sqrt{\frac{2\delta_i R}{r_i L}}\right](\cos k\theta_i)^2 - \frac{\pi}{2}\sqrt{\frac{2LR}{r_i\delta_i}}\exp\left[-2k\sqrt{\frac{2\delta_i R}{r_i L}}\right] + O(1).$$

In the summary, we have

$$\begin{aligned} c_i = {} & \pi\sqrt{\frac{2LR}{r_i\delta_i}}\exp\left[-2k\sqrt{\frac{2\delta_i R}{r_i L}}\right](\cos k\theta_i)^2 + k\alpha_i \\ & + \frac{\pi}{2}\sqrt{\frac{2LR}{r_i\delta_i}}\left(\sqrt{\frac{2k\delta_i}{L\pi}}Li_{1/2}(\exp\left[-2k\frac{\delta_i}{L}\right]) - \exp\left[-2k\sqrt{\frac{2\delta_i R}{r_i L}}\right]\right) + O(1). \end{aligned} \tag{B.37}$$

Now we are going to estimate two integrals in (B.36). We state them as two propositions here and we will prove them separately.

Proposition B.5.1. *The first integral*

$$\int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{kp^2}{1-p^2}d\theta = \frac{\pi}{2}\sqrt{\frac{2LR}{r_i\delta_i}}\sqrt{\frac{2k\delta_i}{L\pi}}Li_{1/2}(\exp\left[-2k\frac{\delta_i}{L}\right]) + O(1),$$

where $p = (1 - d(\theta)/L)^k$.

Proof. Here is the proof for this proposition, it has several steps.

1. First notice that

$$p = (1 - d(\theta)/L)^k, \quad \theta \in (\theta_i - \alpha_i, \theta_i + \alpha_i)$$

where

$$d(\theta_i + \theta) = L - r_i \cos\theta - \sqrt{R^2 - (r_i \sin\theta)^2} = \delta_i + r_i(1 - \cos\theta) + (R - \sqrt{R^2 - (r_i \sin\theta)^2})$$

We can change parameter $\theta = \theta_i + t$ to get

$$\int_{\theta_i - \alpha_i}^{\theta_i + \alpha_i} \frac{kp^2}{1 - p^2} d\theta = \int_{-\alpha_i}^{\alpha_i} \frac{kp^2}{1 - p^2} dt$$

where

$$p^2 = (1 - d(t)/L)^{2k} = e^{2k \ln(1 - d(t)/L)}, \quad t \in (-\alpha_i, \alpha_i)$$

with

$$d(t) = L - r_i \cos t - \sqrt{R^2 - (r_i \sin t)^2} = \delta_i + r_i(1 - \cos t) + (R - \sqrt{R^2 - (r_i \sin t)^2})$$

Define

$$\tilde{d}(t) = \delta_i + \frac{Lr_i}{2R} t^2$$

which is an approximation for $d(t)$. We are going to replace $d(t)$ in the integral by $\tilde{d}(t)$.

2. Then let's look at the following function

$$F(k, x) = \frac{ke^{-2kx}}{1 - e^{-2kx}} = \frac{k}{e^{2kx} - 1}.$$

For fixed k,

$$\left|\frac{\partial F(k,x)}{\partial x}\right| = \left|\frac{-2ke^{2kx}}{(e^{2kx}-1)^2}\right| \leq \frac{2e^{2x}}{(e^{2x}-1)^2} \leq \frac{C}{x^2}, \quad |x| \ll 1.$$

This is because for fixed x,

$$\left|\frac{-2ke^{2kx}}{(e^{2kx}-1)^2}\right|$$

is a decrease function of k.

So we have the following approximation

$$\frac{kp^2}{1-p^2} = \frac{k}{e^{-2k\ln(1-d(t)/L)}-1} = \frac{k}{e^{2k\tilde{d}(t)/L}-1} + \frac{\partial F}{\partial x}(k,\xi)(-\ln(1-d(t)/L)-\tilde{d}(t)/L).$$

where $\xi \approx \tilde{d}(t)/L$. From the analysis above and the approximation in Section B.3, we have

$$\left|\frac{\partial F}{\partial x}(k,\xi)(-\ln(1-d(t)/L)-\tilde{d}(t)/L)\right| \leq C\left|\frac{d(t)/L-\tilde{d}(t)/L}{\xi^2}\right| \leq O(L/R).$$

which means

$$\begin{aligned}\int_{-\alpha_i}^{\alpha_i}\frac{kp^2}{1-p^2}dt &= \int_{-\alpha_i}^{\alpha_i}\frac{kdt}{e^{2k\tilde{d}(t)/L}-1} + O(1)\\ &= \sqrt{\frac{2R\delta_i}{Lr_i}}\int_{-S_i}^{S_i}\frac{kdx}{e^{2k\delta_i/L(1+x^2)}-1} + O(1)\\ &= \frac{1}{2}\sqrt{\frac{2LR}{r_i\delta_i}}\int_{-S_i}^{S_i}\frac{2k\delta_i/Ldx}{e^{2k\delta_i/L(1+x^2)}-1} + O(1)\end{aligned} \tag{B.38}$$

where we changed parameter $t = \sqrt{2R\delta_i/(Lr_i)}x$ and

$$S_i = \frac{\alpha_i}{\sqrt{2R\delta_i/(Lr_i)}} = O(\sqrt{R/\delta_i}) \gg 1.$$

3. Denote $\lambda = 2k\delta_i/L$ here, so λ changes from δ_i/L to infinity as k increase from 1 to ∞. It is also easy to see that $\lambda/(e^{\lambda(1+x^2)} - 1)$ is a decreasing function of λ, hence

$$\frac{\lambda}{e^{\lambda(1+x^2))} - 1} \leq \lim_{\lambda \to 0} \frac{\lambda}{e^{\lambda(1+x^2))} - 1} = \frac{1}{1+x^2}$$

So we have

$$\int_{|x|>S_i} \frac{\lambda}{e^{\lambda(1+x^2))} - 1} \leq \int_{|x|>S_i} \frac{1}{1+x^2} = O(1/S_i) = O(\sqrt{\frac{\delta_i}{R}})$$

In the summary we have

$$\begin{aligned}
\int_{-\alpha_i}^{\alpha_i} \frac{kp^2}{1-p^2} dt &= \frac{1}{2}\sqrt{\frac{2LR}{r_i\delta_i}} \int_{-S_i}^{S_i} \frac{\lambda dx}{e^{\lambda(1+x^2)} - 1} + O(1)] = \frac{1}{2}\sqrt{\frac{2LR}{r_i\delta_i}} \int_{-\infty}^{\infty} \frac{\lambda dx}{e^{\lambda(1+x^2)} - 1} + O(1) \\
&= \frac{1}{2}\sqrt{\frac{2LR}{r_i\delta_i}} \sqrt{\lambda} \int_0^{\infty} \frac{y^{-1/2} dy}{e^y/e^{-\lambda} - 1} + O(1) = \frac{1}{2}\sqrt{\frac{2LR}{r_i\delta_i}} \sqrt{\lambda\pi} Li_{1/2}(e^{-\lambda}) \qquad \text{(B.39)} \\
&= \frac{\pi}{2}\sqrt{\frac{2LR}{r_i\delta_i}} \sqrt{\frac{2k\delta_i}{L\pi}} Li_{1/2}(e^{-2k\delta_i/L}),
\end{aligned}$$

by considering the definition

$$Li_s(z) := \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t/z - 1} dt$$

and the fact that $\Gamma(1/2) = \sqrt{\pi}$. □

Proposition B.5.2. *The second integral*

$$\int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{-2kp^2 \ln p}{(1-p^2)^2} \cos 2k\theta d\theta = \frac{\pi}{2}\sqrt{\frac{2LR}{r_i\delta_i}} \exp\left[-2k\sqrt{\frac{2\delta_i R}{r_i L}}\right] \cos 2k\theta_i + O(1) \quad \text{(B.40)}$$

where $p = (1 - d(\theta)/L)^k$.

Proof. The proof steps will be very similar to the proof in Proposition B.5.1.

1. The first step is still to change parameter $\theta = \theta_i + t$, and use the formula

$$\cos 2k(\theta_i + t) = \cos 2k\theta_i \cos 2kt - \sin 2k\theta_i \sin 2kt.$$

Notice that $\sin 2kt$ is an odd function and the other terms in the integration are even functions. Which means

$$\int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{-2kp^2 \ln p}{(1-p^2)^2} \cos 2k\theta d\theta = \cos 2k\theta_i \int_{-\alpha_i}^{\alpha_i} \frac{-2k^2(1-d(t)/L)^{2k} \ln(1-d(t)/L)}{(1-(1-d(t)/L)^{2k})^2} \cos 2ktdt$$

2. By using the similar method in the proof for Proposition B.5.1, we first use $\tilde{d}(t)$ to approximate $d(t)$ and then change the parameter. We will end up with

$$\begin{aligned}
&\int_{\theta_i-\alpha_i}^{\theta_i+\alpha_i} \frac{-2kp^2 \ln p}{(1-p^2)^2} \cos 2k\theta d\theta \\
&= \frac{1}{2} h_i \cos 2k\theta_i \int_{-S_i}^{S_i} \frac{\lambda^2(1+x^2)e^{\lambda(1+x^2)}}{(e^{\lambda(1+x^2)}-1)^2} \cos(\lambda h_i x) dx + O(1) \\
&= \frac{1}{2} h_i \cos 2k\theta_i \int_{-\infty}^{\infty} \frac{\lambda^2(1+x^2)e^{\lambda(1+x^2)}}{(e^{\lambda(1+x^2)}-1)^2} \cos(\lambda h_i x) dx + O(1) \\
&= \frac{1}{2} h_i \cos 2k\theta_i \Re \left\{ \int_{-\infty}^{\infty} \frac{\lambda^2(1+z^2)e^{\lambda(1+z^2)}}{(e^{\lambda(1+z^2)}-1)^2} e^{i\lambda h_i z} dz \right\} + O(1).
\end{aligned}$$

where $h_i = \sqrt{\frac{2LR}{r_i\delta_i}} = O(\sqrt{\frac{R}{\delta_i}})$. S_i and λ has the exactly same definition as in Proposition B.5.1. Remember that $\lambda = 2k\delta_i/L$.

3. In order to approximate the integral

$$\int_{-\infty}^{\infty} \frac{\lambda^2(1+z^2)e^{\lambda(1+z^2)}}{(e^{\lambda(1+z^2)}-1)^2} e^{i\lambda h_i z} dz$$

with $\lambda > 0$. We consider the following three contours

$$C_1 = \{x + iy : y = 0, \quad -\infty < x < \infty\}$$

$$C_2 = \{x + iy : 2\lambda xy = \pi, \quad 0 < y < \infty\}$$

$$C_3 = \{x + iy : 2\lambda xy = -\pi, \quad 0 < y < \infty\}$$

Actually C_1 is the real axis, C_2 and C_3 are symmetric along the image axis.

$C_1 \cup C_2 \cup (-C_3)$ is a closed contour with counter clockwise direction. Assume the domain between this contour is D. Then the function

$$\frac{\lambda^2(1+z^2)e^{\lambda(1+z^2)}}{(e^{\lambda(1+z^2)}-1)^2}e^{i\lambda h_i z}$$

only has one singular point $z = i$ in D, because $0 \leq \Im(\lambda(1+z^2)) \leq \pi$ in D.

From the residue theorem, we have

$$\int_{C_1\cup C_2\cup(-C_3)} \frac{\lambda^2(1+z^2)e^{\lambda(1+z^2)}}{(e^{\lambda(1+z^2)}-1)^2}e^{i\lambda h_i z}dz = \frac{\pi}{2}e^{-\lambda h_i}.$$

Because

$$\int_{C_1} \frac{\lambda^2(1+z^2)e^{\lambda(1+z^2)}}{(e^{\lambda(1+z^2)}-1)^2}e^{i\lambda h_i z}dz = \int_{-\infty}^{\infty} \frac{\lambda^2(1+z^2)e^{\lambda(1+z^2)}}{(e^{\lambda(1+z^2)}-1)^2}e^{i\lambda h_i z}dz$$

is the integral we want to approximate. Considering the symmetric of C_2 and C_3, it is enough to prove

$$\left|\int_{C_2} \frac{\lambda^2(1+z^2)e^{\lambda(1+z^2)}}{(e^{\lambda(1+z^2)}-1)^2}e^{i\lambda h_i z}dz\right| = O(1).$$

4. On the contour C_2, $2\lambda xy = \pi$, we have

$$\lambda(1+z^2) = \lambda(1+x^2-y^2) + i\pi$$

hence

$$e^{\lambda(1+z^2)} = e^{\lambda(1+x^2-y^2)+i\pi} = -e^{\lambda(1+x^2-y^2)}$$

Also

$$|e^{i\lambda h_i z}| = |e^{i\lambda h_i x} e^{-\lambda h_i y}| = e^{-\lambda h_i y}$$

And we have

$$|dz| = |dx + idy| = \sqrt{1+x'(y)^2}dy = \sqrt{1+(\frac{\pi}{2\lambda})^2\frac{1}{y^4}}dy = \sqrt{1+(\frac{2\lambda}{\pi})^2 x^4}dy.$$

So for large x, we will have

$$\begin{aligned} &\left|\frac{\lambda(1+z^2)e^{\lambda(1+z^2)}}{(e^{\lambda(1+z^2)}-1)^2}dz\right| \le \left|\frac{\lambda(1+z^2)e^{\lambda(1+x^2-y^2)}}{(e^{\lambda(1+x^2-y^2)}+1)^2}dz\right| \le \left|\frac{\lambda(1+z^2)}{e^{\lambda(1+x^2-y^2)}+1}dz\right| \\ &\le C\left|\frac{\lambda x^2 \lambda x^2}{e^{\lambda x^2}}dy\right| \le Cdy \end{aligned} \tag{B.41}$$

And for large y, we will have

$$\begin{aligned} &\left|\frac{\lambda(1+z^2)e^{\lambda(1+z^2)}}{(e^{\lambda(1+z^2)}-1)^2}dz\right| \le \left|\frac{\lambda(1+z^2)e^{\lambda(1+x^2-y^2)}}{(e^{\lambda(1+x^2-y^2)}+1)^2}dz\right| \le C\left|\frac{\lambda(1+z^2)}{e^{\lambda y^2}}dz\right| \\ &\le C\left|\frac{\lambda y^2}{e^{\lambda y^2}}dy\right| \le Cdy \end{aligned} \tag{B.42}$$

Hence we will have

$$\left|\frac{\lambda(1+z^2)e^{\lambda(1+z^2)}}{(e^{\lambda(1+z^2)}-1)^2}dz\right| \leq Cdy$$

for any $x, y > 0$ such that $2\lambda xy = \pi$. Which will give us

$$\left|\int_{C_2} \frac{\lambda^2(1+z^2)e^{\lambda(1+z^2)}}{(e^{\lambda(1+z^2)}-1)^2} e^{i\lambda h_i z} dz\right| \leq \int_0^\infty C\lambda e^{-\lambda h_i y} dy = \frac{C}{h_i} \ll O(1),$$

because

$$h_i = O(\sqrt{\frac{R}{\delta_i}}) \gg 1.$$

In the summary we proved (B.40) for any k.

Bibliography

[1] P.C.N. ALBERTO. On an inverse boundary value problem. *Comp. Appl. Math*, 25(2-3), 2006.

[2] E.S. Bao, Y.Y. Li, and B. Yin. Gradient estimates for the perfect conductivity problem. *Archive for rational mechanics and analysis*, 193(1):195–226, 2009.

[3] L. Berlyand, Y. Gorb, and A. Novikov. Fictitious fluid approach and anomalous blow-up of the dissipation rate in a two-dimensional model of concentrated suspensions. *Archive for rational mechanics and analysis*, 193(3):585–622, 2009.

[4] L. Berlyand and A. Kolpakov. Network approximation in the limit of small interparticle distance of the effective properties of a high-contrast random dispersed composite. *Archive for rational mechanics and analysis*, 159(3):179–227, 2001.

[5] L. Berlyand and A. Novikov. Error of the network approximation for densely packed composites with irregular geometry. *SIAM journal on mathematical analysis*, 34(2):385–408, 2002.

[6] L. Borcea. Asymptotic analysis of quasi-static transport in high contrast conductive media. *SIAM Journal on Applied Mathematics*, pages 597–635, 1998.

[7] L. Borcea, J.G. Berryman, and G.C. Papanicolaou. Matching pursuit for imaging high-contrast conductivity. *Inverse Problems*, 15:811, 1999.

[8] L. Borcea and G.C. Papanicolaou. Network approximation for transport properties of high contrast materials. *SIAM Journal on Applied Mathematics*, pages 501–539, 1998.

[9] Liliana Borcea, Vladimir Druskin, and AV Mamonov. Circular resistor networks for electrical impedance tomography with partial boundary measurements. *Inverse Problems*, 26(4):045010, 2010.

[10] Liliana Borcea, Vladimir Druskin, and Fernando Guevara Vasquez. Electrical impedance tomography with resistor networks. *Inverse Problems*, 24(3):035013, 2008.

[11] Liliana Borcea, Vladimir Druskin, Fernando Guevara Vasquez, and Alexander V Mamonov. Resistor network approaches to electrical impedance tomography. Technical report, 2011.

[12] V.M. Calo, Y. Efendiev, and J. Galvis. Asymptotic expansions for high-contrast elliptic equations.

[13] Tony F Chan and Tarek P Mathew. *Domain decomposition algorithms*. Cambridge Univ Press, 1994.

[14] E. Curtis, E. Mooers, and J. Morrow. Finding the conductors in circular networks from boundary measurements. *Modélisation mathématique et analyse numérique*, 28(7):781–814, 1994.

[15] E.B. Curtis, D. Ingerman, and J.A. Morrow. Circular planar graphs and resistor networks. *Linear algebra and its applications*, 283(1):115–150, 1998.

[16] E.B. Curtis and J.A. Morrow. The dirichlet to neumann map for a resistor network. *SIAM Journal on Applied Mathematics*, 51(4):1011–1029, 1991.

[17] E.B. Curtis and J.A. Morrow. *Inverse problems for electrical networks*, volume 13. World Scientific Publishing Company Incorporated, 2000.

[18] I. Ekeland and R. Temam. Convex analysis and variational problems. 1976.

[19] H. Federer. *Geometric Measure Theory.-Reprint of the 1969 Edition*. Springer, 1996.

[20] J.B. Keller. Conductivity of a medium containing a dense array of perfectly conducting spheres or cylinders or nonconducting cylinders. *Journal of Applied Physics*, 34(4):991–993, 1963.

[21] Sergei Mikhailovich Kozlov. Geometric aspects of averaging. *Russian Mathematical Surveys*, 44(2):91–144, 1989.

[22] A. Novikov. A discrete network approximation for effective conductivity of non-ohmic high-contrast composites. *Communications in Mathematical Sciences*, 7(3):719–740, 2009.

[23] L. Simon. Lectures on geometric measure theory. Centre for Mathematical Analysis, Australian National University, 1984.

[24] J. Sylvester and G. Uhlmann. The dirichlet to neumann map and applications. In *Proceedings of the Conference Inverse problems in partial differential equations (Arcata, 1989), SIAM, Philadelphia*, pages 101–139, 1990.

[25] Andrea Toselli and Olof Widlund. *Domain decomposition methods-algorithms and theory*, volume 34. Springer, 2004.